COMMUTATIVE ALGEBRA HW 4

HOWER, MARTINEZ

Problem 6 Show that if A is a flat $R-$module, then its character module $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ is an injective $R-$module.

Proof: Let $\psi \in \mathrm{Hom}_R(M, N)$ be injective. We must show that

$$\_ \circ \psi \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})), \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})))$$

is surjective. It suffices to show that

$$\_ \circ (\psi \otimes \mathrm{id}_A) : \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R A, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R A, \mathbb{Q}/\mathbb{Z})$$

is surjective. But A is flat, so $\psi \otimes \mathrm{id}_A$ is an injective, $R-$linear map. Furthermore, by Baer's criterion, $\mathbb{Q}/\mathbb{Z}$ is injective so $\_ \circ (\psi \otimes \mathrm{id}_A)$ is a surjective, $\mathbb{Z}-$linear map, as required. $\square$