Problem 5. (Gern, Stanton). *(There is no contravariant version of tensor product.)*

(a) Show that the composite of two contravariant representable functors from the category of R-modules to itself is an additive covariant functor

(b) Show that the composite of two contravariant representable functors from the category of R-modules to itself is not representable when R-is a field.

Solution: (a) Let R be a ring, and let F, G be two contravariant representable functors from the category of R-modules to itself. Then $F = \mathrm{Hom}(\_, D)$ and $G = \mathrm{Hom}(\_, E)$ for some R-modules D and E. Define the functor $H = F \circ G$, let A be an R-module, and let f and g be R-module homomorphisms. Then since F, G are contravariant, we see that

(i) $H(f \circ g) = F(G(f \circ g)) = F(G(g) \circ G(f)) = F(G(f)) \circ F(G(g)) = H(f) \circ H(g)$.

(ii) $H(\mathrm{id}_A) = F(G(\mathrm{id}_A)) = F(\mathrm{id}_{G(A)}) = \mathrm{id}_{F(G(A))} = \mathrm{id}_{H(A)}$.

(iii) $H(\mathrm{dom}(f)) = F(G(\mathrm{dom}(f))) = F(\mathrm{cod}(G(f))) = \mathrm{dom}(F(G(f))) = \mathrm{dom}(H(f))$.

(iv) $H(\mathrm{cod}(f)) = F(G(\mathrm{cod}(f))) = F(\mathrm{dom}(G(f))) = \mathrm{cod}(F(G(f))) = \mathrm{cod}(H(f))$.

so H is a covariant functor. Now let A, B be R-modules and consider the map

$$H : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(H(A), H(B))$$

We see that since F and G are additive functors, then if $f, g \in \mathrm{Hom}_R(A, B)$ then

$$H(f + g) = F(G(f + g)) = F(G(f) + G(g)) = F(G(f)) + F(G(g)) = H(f) + H(g),$$

so H is an abelian group homomorphism, and thus H is an additive covariant functor.

(b) Now let $R = F$ be a field. We will exhibit two contravariant representable functors F and G from the category of F-modules to itself such that $F \circ G$ is not representable. Let $F = Hom(\_, F)$ and $G = Hom(\_, F)$. Suppose that $F \circ G$ is representable. Then by part (a), $F \circ G$ is covariant, so $F \circ G = Hom(W, \_)$ for some vector space W over F. Hence, for all vector spaces V over F,

$$Hom(Hom(V, F), F) \cong Hom(W, V).$$

First of all, say W has dimension 0. Then $Hom(W, V)$ has dimension 0 for all vector spaces V. However, for V of finite positive dimension n, $Hom(Hom(V, F), F) \cong V^{**} \cong V$ has dimension n, so $Hom(Hom(V, F), F)$ is not isomorphic to $Hom(W, F)$.

Now, say W has dimension 1. Then for any infinite dimensional vector space V, V^{**} has dimension of cardinality strictly greater than the cardinality of the dimension of V (see [1], page 435, problem 5). Since the dimension of $Hom(W, V)$ is same as the dimension of V, we see that $Hom(Hom(V, F), F)$ is not isomorphic to $Hom(W, V)$. Thus, W cannot have dimension 1.

Finally, suppose that W has dimension greater than 1 (finite or infinite). Then, taking $V = F$, we see that $Hom(Hom(F, F), F) = F^{**}$ has dimension 1, but $Hom(W, F) = W^*$ has dimension strictly greater than 1. Thus, W cannot have dimension greater than 1.

We conclude that no such W exists, so $Hom(Hom(\_, F), F)$ is not a covariant representable functor.

□

References: 1. Dummitt, D., Foote, R. *Abstract Algebra*, Third Edition. John Wiley and Sons, Inc, 2004.