Assignment IV, Problem 4 Batchelder, Jones, Tuley

4 (a) *Show that* $\mathbb{Q}\left[\sqrt{m}\right] \otimes_{\mathbb{Q}} \mathbb{Q}\left[\sqrt{m}\right] \cong \mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]$ *as* $\mathbb{Q}$*-algebras.*

Solution. Recall the universal property of tensor product: letting ι denote the natural inclusion, any bilinear map φ from $\mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]$ to another $\mathbb{Q}$-module X (in our case, $\mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]$), there exists a unique $\Phi : \mathbb{Q}\left[\sqrt{m}\right] \otimes_{\mathbb{Q}} \mathbb{Q}\left[\sqrt{m}\right]$ so that $\varphi = \Phi \circ \iota$:

$$\begin{tikzcd}
\mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right] \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{Q}\left[\sqrt{m}\right] \otimes_{\mathbb{Q}} \mathbb{Q}\left[\sqrt{m}\right] \arrow[d, dashed, "\Phi"] \\
& \mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]
\end{tikzcd}$$

Define φ by $\varphi : (a, b) \mapsto (ab, a\overline{b})$, where $\overline{\alpha + \beta\sqrt{m}} = \alpha - \beta\sqrt{m}$ denotes the usual conjugate. We claim this is a bilinear map: for $a, b, x, y \in \mathbb{Q}\left[\sqrt{m}\right]$,

$$\begin{aligned}
\varphi(ax + y, b) &= ((ax + y)b, (ax + y)\overline{b}) \\
&= (axb + yb, ax\overline{b} + y\overline{b}) \\
&= a(xb, x\overline{b}) + (yb, y\overline{b}) \\
&= a\varphi(x, b) + \varphi(y, b)
\end{aligned}
\qquad
\begin{aligned}
\varphi(a, bx + y) &= (ab(x + y), a\overline{(bx + y)}) \\
&= (abx + ay, a(\overline{bx} + \overline{y})) \\
&= (abx, ab\overline{x}) + (ay, a\overline{y}) \\
&= b\varphi(a, x) + \varphi(a, y)
\end{aligned}$$

Hence there is a unique Φ which lifts φ.

We now define a basis of $\mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]$ over $\mathbb{Q}$ as a $\mathbb{Q}$-algebra:

$$b_0 = (1, 1) \qquad b_1 = (\sqrt{m}, \sqrt{m}) \qquad b_2 = (\sqrt{m}, -\sqrt{m}) \qquad b_3 = (m, -m).$$

This is a basis: if

$$0 = a(1, 1) + b(\sqrt{m}, \sqrt{m}) + c(\sqrt{m}, -\sqrt{m}) + d(m, -m),$$

then

$$\begin{aligned}
a + md &= 0 & & \\
a - md &= 0 & &\Rightarrow a = 0 \Rightarrow d = 0 \\
b + c &= 0 & & \\
b - c &= 0 & & \\
2b &= 0 & & \\
b &= 0 & &\Rightarrow c = 0.
\end{aligned}$$

A basis of $\mathbb{Q}\left[\sqrt{m}\right] \otimes_{\mathbb{Q}} \mathbb{Q}\left[\sqrt{m}\right]$ is

$$v_0 = 1 \otimes 1 \qquad v_1 = \sqrt{m} \otimes 1 \qquad v_2 = 1 \otimes \sqrt{m} \qquad v_3 = \sqrt{m} \otimes \sqrt{m}$$

by Corollary 19 on pg. 374 of Dummit and Foote.

Let Φ be the induced map $\mathbb{Q}\left[\sqrt{m}\right] \otimes_{\mathbb{Q}} \mathbb{Q}\left[\sqrt{m}\right] \to \mathbb{Q}\left[\sqrt{m}\right] \times \mathbb{Q}\left[\sqrt{m}\right]$; observe that since Φ is a lift of φ, we get $a \otimes b \mapsto (ab, a\overline{b})$. Furthermore, we must send basis to basis, which we observe happens thus:

$$\begin{aligned}
1 \otimes 1 &\mapsto (1, 1) & \sqrt{m} \otimes 1 &\mapsto (\sqrt{m}, \sqrt{m}) \\
1 \otimes \sqrt{m} &\mapsto (\sqrt{m}, -\sqrt{m}) & \sqrt{m} \otimes \sqrt{m} &\mapsto (m, -m).
\end{aligned}$$

Since the image of Φ contains a basis of $\mathbb{Q}\left[\sqrt{m}\right]\times\mathbb{Q}\left[\sqrt{m}\right]$, it is surjective. We now show it is injective: $\mathbb{Q}\left[\sqrt{m}\right]\otimes\mathbb{Q}\left[\sqrt{m}\right]$ is a four-dimensional vector space with the above basis; if a linear combination of those vectors is sent to zero, then the result is some combination of basis vectors in $\mathbb{Q}\left[\sqrt{m}\right]\times\mathbb{Q}\left[\sqrt{m}\right]$, and so is only zero if each coefficient is zero; since Φ is a bilinear map, this means that the preimage must be zero.

Hence Φ is an isomorphism, so $\mathbb{Q}\left[\sqrt{m}\right]\otimes_{\mathbb{Q}}\mathbb{Q}\left[\sqrt{m}\right]\cong\mathbb{Q}\left[\sqrt{m}\right]\times\mathbb{Q}\left[\sqrt{m}\right]$. ∎

(b) *Find the idempotents in* $\mathbb{Q}\left[\sqrt{m}\right]\otimes_{\mathbb{Q}}\mathbb{Q}\left[\sqrt{m}\right]$ *that induce the direct decomposition described in* **(a)**.

Solution. Idempotent elements in $\mathbb{Q}\left[\sqrt{m}\right]\times\mathbb{Q}\left[\sqrt{m}\right]$ are $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$; since Φ is an isomorphism, the pullback of each of those elements is also idempotent. Those pullbacks are:

$$\begin{aligned}
\Phi^{-1}(0,0) &= 0\otimes 0\\
\Phi^{-1}(1,0) &= \frac{1}{2}(1\otimes 1)+\frac{1}{2m}\left(\sqrt{m}\otimes\sqrt{m}\right)\\
\Phi^{-1}(0,1) &= \frac{1}{2}(1\otimes 1)-\frac{1}{2m}\left(\sqrt{m}\otimes\sqrt{m}\right)\\
\Phi^{-1}(1,1) &= \Phi^{-1}(0,1)+\Phi^{-1}(1,0) = (1\otimes 1).
\end{aligned}$$

The two nontrivial idempotents induce the direct decomposition courtesy of homework 1, problem 3. ∎

(c) *Find an idempotent* $e\neq 0,1$ *in* $\mathbb{Q}[\sqrt[3]{2}]\otimes\mathbb{Q}[\sqrt[3]{2}]$.

Solution. One example is $e = \tfrac{2}{3}(1\otimes 1)-\tfrac{1}{6}(\sqrt[3]{2}\otimes\sqrt[3]{4})-\tfrac{1}{6}(\sqrt[3]{2}\otimes\sqrt[3]{4})$:

$$\begin{aligned}
e^2 &= \frac{4}{9}(1\otimes 1)^2-\frac{1}{9}(1\otimes 1)\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)-\frac{1}{9}(1\otimes 1)\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)\\
&\quad-\frac{1}{9}\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)(1\otimes 1)+\frac{1}{36}\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)^2+\frac{1}{36}\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)\\
&\quad-\frac{1}{9}\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)(1\otimes 1)+\frac{1}{36}\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)+\frac{1}{36}\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)^2\\
&= \frac{4}{9}(1\otimes 1)-\frac{1}{9}\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)+\frac{1}{9}\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)-\frac{1}{9}\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)+\frac{1}{36}\left(\sqrt[3]{4}\otimes\sqrt[3]{16}\right)\\
&\quad+\frac{1}{36}\left(\sqrt[3]{8}\otimes\sqrt[3]{8}\right)-\frac{1}{9}\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)+\frac{1}{36}\left(\sqrt[3]{8}\otimes\sqrt[3]{8}\right)+\frac{1}{36}\left(\sqrt[3]{16}\otimes\sqrt[3]{4}\right)\\
&= \left[\frac{4}{9}+\frac{4}{36}+\frac{4}{36}\right](1\otimes 1)+\left[-\frac{1}{9}-\frac{1}{9}+\frac{2}{36}\right]\left(\sqrt[3]{2}\otimes\sqrt[3]{4}\right)+\left[-\frac{1}{9}-\frac{1}{9}+\frac{2}{36}\right]\left(\sqrt[3]{4}\otimes\sqrt[3]{2}\right)\\
&= e.
\end{aligned}$$

This example was found by observing the structure of the idempotents in the previous question, then generalizing the structure and searching likely candidates until a suitable element was found. ∎