COMMUTATIVE ALGEBRA HW 4

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Problem 3 Let $\mathbb{F}$ be a field. Suppose that A and B are F-algebras and that $B = \mathbb{F}[b]$ is generated as an $\mathbb{F}$-algebra by a single element $b \in B$.

(1) Show that $A \otimes_{\mathbb{F}} B \cong A[x]/\min_{b,\mathbb{F}}(x)$.
(2) Restrict now to the case where A and B are fields. Give an example where $A \otimes_{\mathbb{F}} B$ has nonzero nilpotent elements, and another example where $A \otimes_{\mathbb{F}} B$ $(\neq 0)$ has no nonzero nilpotent elements.

Solution

(1) An $\mathbb{F}$-algebra homomorphism $\varphi\colon \mathbb{F}[x] \to \mathbb{F}[b]$ is determined by $\varphi(x)$ since $\mathbb{F}[x]$ is freely generated by x as an $\mathbb{F}$-algebra. The map $x \mapsto b$ extends uniquely to such an $\mathbb{F}$-algebra homomorphism φ. Since $\mathbb{F}[x]$ is a principal ideal domain, there exists a unique monic polynomial that generates the ideal $\ker\varphi \lhd \mathbb{F}[x]$ (we consider the zero polynomial to be monic). We define $\min_{b,\mathbb{F}}(x)$ to be this unique monic polynomial. Henceforth, let

$$m(x) = \sum_{k=0}^{n} m_k x^k$$

be the polynomial $\min_{b,\mathbb{F}}(x)$.

Now let $\alpha \otimes \beta$ be a simple tensor in $A \otimes_{\mathbb{F}} B$. We define a **normal form** for any simple tensor in $\alpha \otimes \beta \in A \otimes B$ as follows. We can represent this simple tensor uniquely as $\alpha \otimes \beta = c\alpha \otimes p(b)$ where $c \in \mathbb{F}$ and $p(x)$ is monic and of minimal degree. If $\alpha \otimes \beta$ is zero, set $c = 0$ and $p(x) = 0$. Otherwise we obtain c and $p(x)$ by viewing $\beta = q(b)$ where $[q(x)]$ is an element in $\mathbb{F}[x]/(m(x))$ and choosing a representative $q'(x)$ of this class of minimal degree. Then $q'(x)$ can be expressed uniquely as $cp(x)$ where $p(x)$ is monic and $c \in \mathbb{F}$. It is clear that $cp(b) = q(b) = \beta$ since $cp(x) - q(x) \in (m(x))$ and hence $cp(b) - q(b) = 0$.

Then we define a map $f\colon A \times B \to A[x]/m(x)$ as follows. Let $(\alpha, \beta) \in A \times B$. Let $c\alpha \otimes p(b)$ be the normal form of the simple tensor $\alpha \otimes \beta$. Then we define

$$f(\alpha, \beta) = [c\alpha p(x)]$$

where $[c\alpha p(x)]$ is the equivalence class of $c\alpha p(x)$. We will show that this map is $\mathbb{F}$-balanced. Let $\alpha_1, \alpha_2 \in A$, $\beta_1, \beta_2 \in B$. Consider the simple tensor $\alpha_1 \otimes (\beta_1 + \beta_2)$ with normal form $c\alpha_1 \otimes p(b)$ where $c \in \mathbb{F}$. Now let $c_1\alpha_1 \otimes p_1(b)$ and $c_2\alpha_1 \otimes p_2(b)$ be the normal forms of $\alpha_1 \otimes \beta_1$ and $\alpha_1 \otimes \beta_2$, respectively. Thus $cp(x) \equiv c_1p_1(x) + c_2p_2(x) \pmod{m(x)}$, so

$$\begin{aligned} f(\alpha_1, \beta_1) + f(\alpha_1, \beta_2) &= [c_1\alpha_1 p_1(x)] + [c_2\alpha_1 p_2(x)] \\ &= [\alpha_1 (c_1p_1(x) + c_2p_2(x))] \\ &= [c\alpha_1 p(x)] \\ &= f(\alpha_1, \beta_1 + \beta_2). \end{aligned}$$

Now let $c\alpha_1 \otimes p(b)$ and $c\alpha_2 \otimes p(b)$ be the normal form of $\alpha_1 \otimes \beta_1$ and $\alpha_2 \otimes \beta_1$, respectively. Then $(c\alpha_1 + c\alpha_2) \otimes p(b)$ is the normal form of $(\alpha_1 + \alpha_2) \otimes \beta$ and we have

$$f(\alpha_1 + \alpha_2, \beta) = [(c\alpha_1 + c\alpha_2)p(x)] = [c\alpha_1 p(x)] + [c\alpha_2 p(x)] = f(\alpha_1, \beta_1) + f(\alpha_2, \beta_1).$$

Our definition of f implies that if $c \in \mathbb{F}$ then

$$f(\alpha_1 c, \beta_1) = f(\alpha_1, c\beta_1)$$

since the corresponding simple tensors have the same normal form. Then f is $\mathbb{F}$-balanced, so by the universal property of tensor product f extends uniquely to an $\mathbb{F}$-algebra homomorphism $\varphi\colon A \otimes B \to A[x]/m(x)$.

$$\begin{tikzcd}
A \times B \arrow[r, "\subseteq"] \arrow[dr, "f"'] & A \otimes B \arrow[d, dotted, "\varphi"] \\
 & A[x]/m(x)
\end{tikzcd}$$

We note that $A[x]$ is generated by elements of the form αx^n where $\alpha \in A$ and $n \geq 0$. Then $A[x]/m(x)$ is generated by the corresponding equivalence classes. Since

$$\varphi(\alpha \otimes b^n) = [\alpha x^n]$$

it follows that φ maps $A \otimes B$ onto a generating set for $A[x]$ and is therefore surjective.

Now we define a map $\rho : A[x] \to A \otimes B$ by setting $a_n x^n + \ldots + a_0 \mapsto a_n \otimes b^n + \ldots + a_0 \otimes 1$. Now if $\alpha \otimes \beta$ is any nonzero simple tensor then $\alpha \otimes \beta$ has some normal form $c\alpha \otimes p(b)$, hence $\rho(c\alpha p(x)) = c\alpha \otimes p(b) = \alpha \otimes \beta$, so ρ is surjective. Note that if $c\alpha \otimes p(b)$ is the normal form of any simple tensor then

$$\rho(\varphi(c\alpha \otimes p(b))) = \rho(c\alpha p(x)) = c\alpha \otimes p(b).$$

Now because φ is surjective and ρ is its left inverse, φ and ρ are inverses of each other, thus φ is an isomorphism.

(2) First consider the case $A = B = \mathbb{F}$, for any field $\mathbb{F}$. Note that $\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}$ via $a \otimes b \mapsto a \cdot b$. Fields do not have nonzero nilpotent elements.

Now let $\mathbb{F}_p$ be the field with p elements, for some prime, p, and let $K = \mathbb{F}_p[k]$ be a transcendental extension of $\mathbb{F}_p$. Now let α be a root of the polynomial $x^p - k$. Thus $K[\alpha]$ is a free K-module of rank p, with basis $1, \alpha, \ldots, \alpha^{p-1}$. Thus $K[\alpha] \otimes_K K[\alpha]$ is free of rank p^2. The set of simple tensors $b_i \otimes b_j$ for any two basis elements b_i, b_j of $K[\alpha]$ are clearly a generating set for $K[\alpha] \otimes K[\alpha]$, and by comparing their number to the dimension we see they must form a basis. Because the basis is linearly independent, we must have that $a := \alpha \otimes 1 - 1 \otimes \alpha$ is a nonzero element of $K[\alpha] \otimes_K K[\alpha]$. However,

$$\begin{aligned}
a^p &= (\alpha \otimes 1 - 1 \otimes \alpha)^p \\
&= \alpha^p \otimes 1^p - 1^p \otimes \alpha^p \\
&= k \otimes 1 - 1 \otimes k \\
&= 1 \otimes k - 1 \otimes k \\
&= 1 \otimes 0,
\end{aligned}$$

thus a is a nonzero nilpotent of $A \otimes B$.