COMMUTATIVE ALGEBRA: HOMEWORK 4

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2) Let A and B be R-algebras, and define $i_A : A \to A \otimes_R B : a \mapsto a \otimes 1$ and $i_B : B \to A \otimes_R B : b \mapsto 1 \otimes b$

(a) Show that $(A \otimes B, i_A, i_B)$ is a coproduct of A and B in the category of R-algebras.

(b) Show that if A and B are the R-algebras presented by generators and relations as $A = \langle G \mid S \rangle$ and $B = \langle H \mid T \rangle$, then $A \otimes_R B \cong \langle G \sqcup H \mid S \sqcup T \rangle$.

Solution

(a) *Proof.* For $a, b \in A$ and $r, t \in R$, we have

$$\begin{aligned} i_A(ra + b) &= (ra + b) \otimes 1 = r(a \otimes 1) + (b \otimes 1) = ri_A(a) + i_A(b), \\ i_A(ab) &= (ab) \otimes 1 = (a \otimes 1)(b \otimes 1) = i_A(a)i_A(b), \text{ and} \\ i_A(1) &= 1 \otimes 1 = 1. \end{aligned}$$

Hence i_A is a morphism of R-algebras. A similar argument shows that i_B is also a morphism. We must show that if C is an R-algebra with morphisms $f : A \to C$ and $g : B \to C$ then there is unique morphism $f \amalg g : A \otimes B \to C$ such that the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A \otimes_R B \arrow[u, dotted, "f \amalg g"'] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

commutes. Define a linear map $h : A \otimes B \to C$ by $h(a \otimes b) = f(a)g(b)$. We will show that this map determines an R-algebra morphism. Let $r \in R$ and $a \otimes b, a' \otimes b' \in A \otimes B$. Then

$$\begin{aligned} h(r(a \otimes b)(a' \otimes b')) &= h(r(aa') \otimes (bb')) = f(raa')g(bb') = rf(a)f(a')g(b)g(b') \\ &= rf(a)g(b)f(a')g(b') = rh(a \otimes b)h(a' \otimes b'), \end{aligned}$$

and

$$h(1) = h(1 \otimes 1) = f(1)g(1) = (1)(1) = 1$$

(f and g are R-algebra homomorphism and hence map 1 to 1), so h is a morphism of R-algebras.

We will now show that the diagram commutes if and only if $f \amalg g = h$, proving that h satisfies the definition of the coproduct map and that it is unique. The diagram commutes if and only if $(f \amalg g)i_A = f$ and $(f \amalg g)i_B = g$. Hence

$$\begin{aligned} f \amalg g(a \otimes b) &= f \amalg g((a \otimes 1)(1 \otimes b)) = f \amalg g(a \otimes 1) f \amalg g(1 \otimes b) \\ &= (f \amalg g)i_A(a)(f \amalg g)i_B(b) = f(a)g(b). \end{aligned}$$

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Thus $f \amalg g(a \otimes b) = f(a)g(b) = h(a \otimes b)$. Since the simple tensors generate the R-algebra $A \otimes B$, we have $h = f \amalg g$. □

(b) *Proof.* Since i_A is an R-algebra morphism, elements $i_A(G) = \{g \otimes 1 \mid g \in G\}$ satisfy all of the identities in $A \otimes B$ that G satisfied in A. The same is true for i_B, so the mapping

$$\begin{aligned} \varphi : \langle G \sqcup H \mid S \sqcup T \rangle &\to A \otimes_R B \\ g &\mapsto g \otimes 1 \\ h &\mapsto 1 \otimes h \end{aligned}$$

is compatible with the relations $S \sqcup T$. Therefore by the universal property of presentations φ determines a morphism of R-algebras.

Define mappings

$$\begin{aligned} \alpha : G &\to \langle G \sqcup H \mid S \sqcup T \rangle : g \mapsto g, \\ \beta : H &\to \langle G \sqcup H \mid S \sqcup T \rangle : h \mapsto h. \end{aligned}$$

Since $S \subseteq S \sqcup T$, the relations that G satisfies in $\langle G \mid S \rangle$ are also satisfied in $\langle G \sqcup H \mid S \sqcup T \rangle$. Therefore α is compatible with the relations S, so by the universal property α extends to a unique morphism from A to $\langle G \sqcup H \mid S \sqcup T \rangle$. A similar argument shows that β also extends to a unique morphism from B to $\langle G \sqcup H \mid S \sqcup T \rangle$. By part (a) above, α and β induce a unique morphism $\alpha \amalg \beta : A \otimes B \to \langle G \sqcup H \mid S \sqcup T \rangle$ defined by $\alpha \amalg \beta(g \otimes 1) = g$ and $\alpha \amalg \beta(1 \otimes h) = h$. The claim is that $\varphi^{-1} = \alpha \amalg \beta$. For $g \in G$ and $h \in H$, we have

$$\varphi(\alpha \amalg \beta)(g \otimes h) = \varphi(gh) = g \otimes h,$$

and

$$(\alpha \amalg \beta)\varphi(gh) = \alpha \amalg \beta(g \otimes h) = gh.$$

Since $\langle G \mid S \rangle$ presents A and $\langle H \mid T \rangle$ presents B, and the simple tensors generate the algebra $A \otimes B$, this shows that $\alpha \amalg \beta$ is the inverse of φ. Therefore φ and $\alpha \amalg \beta$ are isomorphisms, and $A \otimes_R B \cong \langle G \sqcup H \mid S \sqcup T \rangle$. □