COMMUTATIVE ALGEBRA: HOMEWORK 3

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(7) An algebra is said to be *Hopfian* if every surjective endomorphism is an automorphism.

(a) Prove that every finitely generated module over a commutative ring is Hopfian.

(b) Give an example of a finitely generated module over a noncommutative ring that is not Hopfian. Explain why no such module can be Noetherian.

SOLUTION

Proof. (a) Suppose that M is a finitely generated R module over a commutative ring R and that $\varphi : M \to M$ is surjective. M can be made into an $R[\varphi]$ module, where $R[\varphi]$ are polynomials over R in the symbol φ, as follows: Given $p(\varphi) = \sum_{i=0}^{n} r_i \varphi^i$ and $m \in M$, we let $p(\varphi) \cdot m = \sum_{i=0}^{n} r_i \varphi^i(m)$. This action satisfies all of the module identities, as is easily verified. Let I be the ideal generated in $R[\varphi]$ by φ. Then as φ is surjective, we have that $IM = M$. By the strong form of Nakayama's Lemma, there exists $p(\varphi) \in I$ such that $\lambda_{p(\varphi)} : M \to M$ is the identity function. Note that $p(\varphi)$ has no constant term. Suppose that $m \in ker(\varphi)$. Then $\lambda_{p(\varphi)}(m) = 0$, so $m = 0$. Therefore φ is injective, and thus is an automorphism.

(b) Let $S = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ and let $N = \begin{pmatrix} \mathbb{Q}/\mathbb{Z} \\ \mathbb{Q} \end{pmatrix}$. It's easy to check that N is a left S-module under the ring action:

$$\begin{pmatrix} m & r \\ 0 & s \end{pmatrix} \begin{pmatrix} p + \mathbb{Z} \\ q \end{pmatrix} = \begin{pmatrix} mp + rq + \mathbb{Z} \\ sq \end{pmatrix}.$$

Also N is generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Further, multiplication by 2 clearly gives a surjective endomorphism on N and has a non trivial kernel of $\left\{ \begin{pmatrix} 0 + \mathbb{Z} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + \mathbb{Z} \\ 0 \end{pmatrix} \right\}$. Therefore, N is not Hopfian. This example is not Noetherian. Indeed, any Noetherian module, M over any (not necessarily commutative) ring R is Hopfian, for if φ is a surjective endomorphism on M, then so is φ^i for all positive integers i. We also have

$$\ker \varphi \subseteq \ker \varphi^2 \subseteq \ker \varphi^3 \subseteq \ldots,$$

and so, since M is Noetherian, $\ker \varphi^i = \ker \varphi^{i+1}$ for some positive integer i. Let $m \in \ker \varphi$. Since φ is surjective, so is φ^i, and there exists some $m' \in M$ such that $m = \varphi^i(m')$. Hence, $0 = \varphi(m) = \varphi^{i+1}(m')$, so $m' \in \ker \varphi^{i+1} = \ker \varphi^i$. Therefore $0 = \varphi^i(m') = m$, and φ is an automorphism.

□

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