Exercise 6 *(Chriestenson, Keller) Suppose that* $(R, \mathfrak{m})$ *is a local ring with maximal ideal* $\mathfrak{m}$*, and that* M *is a finitely generated* R*-module.*

(1) *Show that a subset* $F \subseteq M$ *is a generating set iff* $F/\mathfrak{m}M = \{f + \mathfrak{m}M \in M/\mathfrak{m}M | f \in F\}$ *is a generating set for the* $R/\mathfrak{m}$*-vector space* $M/\mathfrak{m}M$*. Conclude that all minimal generating sets for* M *have the same size.*
(2) *Show that a homomorphism* $\phi : M \to N$ *between finitely generated* R*-modules is surjective iff the induced map* $\phi_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ *is surjective.*

Proof: We note that since $\mathfrak{m}$ annihilates $M/\mathfrak{m}M$ we can equip $M/\mathfrak{m}M$ with a well defined $R/\mathfrak{m}$-vector space structure given by $(r + \mathfrak{m})(a+\mathfrak{m}M) = (ra+\mathfrak{m}M)$ for all $r+\mathfrak{m} \in R/\mathfrak{m}$ and all $a+\mathfrak{m}M \in M/\mathfrak{m}M$. Let $F \subseteq M$ be a set that generates M. For any $a + \mathfrak{m}M \in M/\mathfrak{m}M$ we have that there is a $n \in \mathbb{Z}_{\geq 0}$ such that for each $1 \leq i \leq n$ there are $r_i \in R$, $f_i \in F$ such that $a = r_1 f_1 + r_2 f_2 + ... + r_n f_n$. Thus $a + \mathfrak{m}M = a_1 f_1 + ... + a_n f_n + \mathfrak{m}M = (r_1 + \mathfrak{m})(f_1 + \mathfrak{m}M) + (r_2 + \mathfrak{m})(f_2 + \mathfrak{m}M) + ... + (r_n + \mathfrak{m})(f_n + \mathfrak{m}M)$. And thus the set $F/\mathfrak{m}M = \{f + \mathfrak{m}M | f \in F\}$ generates $M/\mathfrak{m}M$ as an $R/\mathfrak{m}$-vector space.

Conversely let $F/\mathfrak{m}M$ be a generating set of $M/\mathfrak{m}M$. Let $N =< F >$ be the R-subbodule of M generated by F. Then the composition of the inclusion map $N \to M$ and the projection $M \to M/\mathfrak{m}M$ gives us a surjective R-module homomorphism $N \to M/\mathfrak{m}M$. Thus we have that $N + \mathfrak{m}M = M$. We have that M is finitely generated and $\mathfrak{m} = \mathrm{Rad}\, R$. Thus by the weak form of Nakayama's Lemma we see that $M = N$. Therefore the elements $\{f \in M | f + \mathfrak{m}M \in M/\mathfrak{m}M\}$ generate M.

Let F be any generating set of M that is minimal. Then the set $F/\mathfrak{m}M$ generates $M/\mathfrak{m}M$ as an $R/\mathfrak{m}$ vector space. Since $M/\mathfrak{m}M$ is finitely generated it is finite dimensional. Let $n = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$ and let $F' = \{f_1, f_2, ..., f_n\} \subseteq F$ such that the set $F'/\mathfrak{m}M$ is an $R/\mathfrak{m}$-basis in $M/\mathfrak{m}M$. Then since $F'/\mathfrak{m}M$ generates $M/\mathfrak{m}M$ we have that F' must generate M. Because F was chosen minimally and $F' \subseteq F$ we have $F = F'$. Hence every minimal generating set of M has $\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$ generators.

Let M and N be two finitely generated R-modules, and $\phi : M \to N$ an R-module homomorphism. We note that if $x \in \mathfrak{m}M$ then it is of the form $x = rm$ for some $r \in \mathfrak{m}$ and $m \in M$. Thus $\phi(x) = \phi(rm) = r\phi(m) \in \mathfrak{m}N$, and so $\phi(\mathfrak{m}M) \subseteq \mathfrak{m}N$. This implies that the induced map $\phi_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N : m + \mathfrak{m}M \mapsto \phi(m) + \mathfrak{m}N$ is a well defined map. The induced map is clearly a homomorphism since ϕ is. If ϕ is surjective then for each $x+\mathfrak{m}N \in N/\mathfrak{m}N$ let $a \in M$ such that $\phi(a) = x$.

Then we have that $\phi_{\mathfrak{m}}(a + \mathfrak{m}M) = \phi(a) + \mathfrak{m}N = x + \mathfrak{m}N$. Therefore we have that $\phi_{\mathfrak{m}}$ is surjective.

Conversely if $\phi_{\mathfrak{m}}$ is surjective let $P \subseteq N$ be the image of ϕ. Just as above the composition of the injection $P \to N$ and the projection $N/\mathfrak{m}N$ is surjective since $\phi_{\mathfrak{m}}$ is surjective. Thus we have that $P + \mathfrak{m}N = N$. Hence by the same argument as above $P = N$ and ϕ is surjective.