COMMUTATIVE ALGEBRA: HOMEWORK 3

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4) (a) Let M be a finitely generated faithful R-module. Show that M is Noetherian if and only if R is a Noetherian ring.

(b) Suppose that R is a subring of S and that S is finitely generated as an R-module. Show that R is Noetherian if and only if S is Noetherian.

Solution

(a) *Proof.* Suppose that M has generating set $\{m_1, \dots, m_k\}$. Define an R-module homomorphism

$$\varphi : R \to \bigoplus_1^k M : 1 \mapsto (m_1, m_2, \dots, m_k).$$

Suppose that $\varphi(r) = 0$. Then $0 = \varphi(r) = (rm_1, rm_2, \dots, rm_k)$, and so $rm_i = 0$ for all $1 \le i \le k$. Since the multiplicative action of R on M is faithful, this implies that $r = 0$. Hence $\ker \varphi = 0$, and we have a short exact sequence

$$0 \to R \xrightarrow{\varphi} \bigoplus_1^k M \to M \to 0$$

If M is Noetherian then $\bigoplus M$ is also Noetherian. This, in turn, implies that R is Noetherian.

Suppose that R is Noetherian and that M is finitely generated with generating set G, $|G| = m$. We have the following short exact sequence

$$0 \to R \to \bigoplus_{i=1}^m R \to R \to 0.$$

R is Noetherian, so $\bigoplus_{i=1}^m R$ is Noetherian as well. Let

$$\mathrm{Free}_R(G) = \bigoplus_{g \in G} Rg \cong \bigoplus_{i=1}^m R$$

be the free R-module with generators taken from G. Then there is a surjective homomorphism $\mathrm{Free}_R(G) \to M$. Hence $\mathrm{SubMod}(M)$ is isomorphic to a sublattice of $\mathrm{SubMod}(\mathrm{Free}_R(G))$. $\mathrm{SubMod}(\mathrm{Free}_R(G))$ satisfies the ascending chain condition, so $\mathrm{SubMod}(M)$ does as well. □

(b) *Proof.* We will first show that the action of R on S is faithful. Suppose that there is $r \in R$ such that $rs = 0$ for all $s \in S$. Letting $s = 1 - r$, we have $r(1-r) = 0$. Thus $r = r^2$. However, $r = r^2 = rr = 0$, so $r = 0$. Hence the action of R on S is faithful, so S is a finitely generated faithful R-module. By part (a) above, R is Noetherian if and only if S is Noetherian as an R-module. □

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