Problem 3. (Gern, Hower). *(a) Suppose that a and b are elements of a modular lattice L. Show that the canonical isomorphisms*

$$\begin{aligned} \alpha &: [a \cap b, a] \to [b, a+b] &&: x \mapsto x + b \\ \beta &: [b, a+b] \to [a \cap b, a] &&: y \mapsto y \cap a \end{aligned}$$

map compact elements to compact elements. Deduce that M is finitely generated over $M \cap N$ iff $M + N$ is finitely generated over N.

(b) Prove that if $M \cap N$ and $M + N$ are both finitely generated, then so are M and N.

Solution: Let a, b be elements of a modular lattice L. Define α and β as above. Since L is modular, we know that α, β are mutually inverse isomorphisms.

(a) Let $c \in [a \cap b, a]$ be compact in $[a \cap b, a]$. Now consider $\alpha(c) = c + b$. Let $Z \subset L$ be a subset of $[b, a+b]$ such that $c + b \leq \vee Z$. Then since L is modular, and since $\beta(x) = x \cap a = \alpha^{-1}(x)$ is an isomorphism, we get

$$\begin{aligned} c+b &\leq \vee Z \\ \beta(c+b) &\leq \beta[\vee Z] \\ (c+b) \cap a &\leq (\vee Z) \cap a \\ (c \cap a) + (b \cap a) &\leq \vee_{z \in Z}(z \cap a) \\ c &\leq \vee_{z \in Z}(z \cap a). \end{aligned}$$

Then since c is compact, there exists some finite $Z_0 \subset Z$ such that $c \leq \vee_{z \in Z_0}(z \cap a)$. Then since L is modular, and since $\alpha(x) = x + b = \beta^{-1}(x)$ is an isomorphism, we get

$$\begin{aligned} c &\leq \vee_{z \in Z_0}(z \cap a) \\ \alpha(c) &\leq \alpha[\vee_{z \in Z_0}(z \cap a)] \\ c+b &\leq (\vee_{z \in Z_0}(z \cap a)) + b \\ c+b &\leq \vee_{z \in Z_0}((z \cap a) + b) \\ c+b &\leq \vee_{z \in Z_0}((z+b) \cap (a+b)) \\ c+b &\leq \vee_{z \in Z_0}(z), \end{aligned}$$

so there exists some finite subset $Z_0 \subset Z$ that covers $c + b$, thus $c + b$ is compact.

Now suppose that $d \in [b, a+b]$ is compact. Consider $\beta(d) = d \cap a$. Let $Z \subset L$ be a subset of $[a \cap b, a]$ such that $d \cap a \leq \vee Z$. Then since L is modular, and since $\alpha(x) = x + b = \beta^{-1}(x)$ is an isomorphism, we get

$$\begin{aligned} d \cap a &\leq \vee Z \\ \alpha(d \cap a) &\leq \alpha[\vee Z] \\ (d \cap a) + b &\leq (\vee Z) + b \\ (d+b) \cap (a+b) &\leq \vee_{z \in Z}(z+b) \\ d &\leq \vee_{z \in Z}(z+b). \end{aligned}$$

Then since d is compact, there exists some finite $Z_0 \subset Z$ such that $d \leq \vee_{z \in Z_0}(z + b)$. Then since L is modular, and since $\beta(x) = x \cap a = \alpha^{-1}(x)$ is an isomorphism, we get

$$
\begin{aligned}
d &\leq \vee_{z\in Z_0}(z+b) \\
\beta(d) &\leq \beta[\vee_{z\in Z_0}(z+b)] \\
d\cap a &\leq (\vee_{z\in Z_0}(z+b))\cap a \\
d\cap a &\leq \vee_{z\in Z_0}((z+b)\cap a) \\
d\cap a &\leq \vee_{z\in Z_0}((z\cap a)+(b\cap a)) \\
d\cap a &\leq \vee_{z\in Z_0}(z),
\end{aligned}
$$

so there exists some finite subset $Z_0 \subset Z$ that covers $d \cap a$, thus $d \cap a$ is compact. Thus, α and β map compact elements to compact elements.

Now let R be a ring, and let L be the lattice of R-modules. We will show that a module M is compact in L exactly when M is finitely generated. Suppose that M is compact. Let $A \subset M$ generate M, so $M = RA$. Then $M = \vee_{a\in A}(Ra)$, so since M is compact there exist some finite $A_0 \subset A$ such that $M = \vee_{a\in A_0}(Ra)$, so $M = RA_0$, and thus M is finitely generated. Now suppose that M is finitely generated by a set $A = \{a_1, \ldots, a_n\}$. Let $Z \subset L$ such that $M \leq \vee Z = \vee_{z\in Z} Rz$. Then each $a_i \in A$ lies in some Rz_i, so $M \leq \vee_{i=1}^{n} Rz_i = R\{z_1, \ldots, z_n\}$. Thus, a finite subset of Z generates M, so M is compact.

Now let M and N be R-modules. Suppose that M is finitely generated over $M \cap N$. Then $M/(M \cap N)$ is finitely generated, and thus compact in its own submodule lattice, so M is compact in $[M \cap N, M]$. Then from the first part of the problem, $M + N$ is compact in $[N, M+N]$, so $(M+N)/N$ is compact and thus finitely generated, therefore $M+N$ is finitely generated over N. We see that all of the above statements are if and only if, so the converse also holds, and thus M is finitely generated over $M \cap N$ iff $M + N$ is finitely generated over N.

(b) Let M, N be given as above, and suppose that $M + N$ and $M \cap N$ are finitely generated. Since $M + N$ is finitely generated over N, the compactness-preserving property of the lattice isomorphisms from part (a) give us that M is finitely generated over $M \cap N$. This means that $\frac{M}{M\cap N}$ is finitely generated.

Let $\{m_i + M \cap N : 1 \leq i \leq n\}$ be a generating set for $\frac{M}{M\cap N}$, and let $\{x_j : 1 \leq j \leq l\}$ be a generating set for $M \cap N$. Let $m \in M$. Then there exist $r_i \in R$, $1 \leq i \leq n$ such that $m - \sum r_i m_i = x \in M \cap N$. Since $M \cap N$ is finitely generated, there exist $s_j \in R$, $1 \leq j \leq l$, such that $x = \sum s_j x_j$. Putting these two equations together gives us that $m = \sum r_i m_i + \sum s_j x_j$. Thus M is generated by the finite set $\{r_i : 1 \leq i \leq n\} \cup \{s_j : 1 \leq j \leq l\}$. By symmetry, N is also finitely generated.

□