Problem 1 (Li, Selker, Stanton)**.** *Here we consider* $\mathrm{Spec}(R)$ *as a topological space (the primes equipped with the Zariski topology), and as an ordered set (the primes equipped with the inclusion order).*

(a) The inclusion order on the prime ideals can be recovered from the topology of $\mathrm{Spec}(R)$*.*

(b) Conversely, if R is a Noetherian ring, then the topology of $\mathrm{Spec}(R)$ *can be determined from the inclusion order on the prime ideals.*

(c) If R is not Noetherian, then the topology of $\mathrm{Spec}(R)$ *may not be recoverable from the inclusion order on the primes.*

Proof. (a) Note that for prime ideals $\mathfrak{a}, \mathfrak{b}$, we have $\mathfrak{a} \subseteq \mathfrak{b}$ iff $\mathrm{cl}(\mathfrak{b}) \subseteq \mathrm{cl}(\mathfrak{a})$, so the inclusion order can always be obtained from the topology.

(b) Let $(\mathrm{Spec}(R), \subseteq)$ be the inclusion ordering of the primes. If $\mathfrak{a} \in \mathrm{Spec}(R)$, the closure of $\mathfrak{a}$ is defined by $\mathrm{cl}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{a} \subseteq \mathfrak{p}\}$. Define $F = \{\mathrm{cl}(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(R)\}$, and let $\overline{F}$ be the closure of F under finite unions. Then $\overline{F}$ is a collection of closed sets in $\mathrm{Spec}(R)$. We claim that if R is Noetherian, every closed set of $\mathrm{Spec}(R)$ is a member of $\overline{F}$. To see this, let C be a closed set of $\mathrm{Spec}(R)$. By definition of the topology, there is some $\mathfrak{a} \lhd R$ such that $C = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{a} \subseteq \mathfrak{p}\}$. By (the proof of) problem 5 on assignment 2, if $\mathfrak{p}$ is a prime containing $\mathfrak{a}$, then there is a prime $\hat{\mathfrak{p}}$ contained in $\mathfrak{p}$ that is minimal among primes containing $\mathfrak{a}$. Thus we may restrict our attention to these minimal primes, i.e.

$$C = \bigcup_{\mathfrak{p} \supseteq \mathfrak{a}} \{\mathfrak{q} \in \mathrm{Spec}(R) : \hat{\mathfrak{p}} \subseteq \mathfrak{q}\} = \bigcup_{\mathfrak{p} \supseteq \mathfrak{a}} \mathrm{cl}(\hat{\mathfrak{p}}).$$

By problem 6 on assignment 2, if R is Noetherian this is a finite union, thus the closed sets C are exactly the finitely generated order filters in $(\mathrm{Spec}(R) \subseteq)$, so they are all contained in $\overline{F}$ as claimed.

(c) We give an example to show that the topology of $\mathrm{Spec}(R)$ may not be recoverable from the inclusion ordering of $\mathrm{Spec}(R)$ in case R is not Noetherian. Consider the ring $R := \prod_{i \in \omega} \mathbb{F}_2$. Thus, we may identify elements of R with functions $f : \omega \to 2$. Note that $f^2 = f$ for all $f \in R$. Thus for any $f \in R$ we have $f(1-f) = f - f^2 = f - f = 0$. Thus if $\mathfrak{p}$ is a prime ideal, then $f(1-f) = 0 \in \mathfrak{p}$, so either f or $(1-f)$ is in $\mathfrak{p}$. Thus each prime has index 2 and is therefore is maximal. Thus the ordering on $\mathrm{Spec}(R)$ is the trivial partial-order where $\mathfrak{a} \leq \mathfrak{b} \Leftrightarrow \mathfrak{a} = \mathfrak{b}$. Note that there are primes that arise as kernels of projections π onto single coordinates, yielding by the F.I.T. $R \cong \ker(\pi) \times \mathbb{F}_2$. By exercise 22 on page 13 of the text, this yields a clopen singleton $\{\ker(\pi)\}$. Now if $(\mathrm{Spec}(R), \subseteq)$ were to generate the topology on $\mathrm{Spec}(R)$ then any order-automorphism would merely permute the closed sets, and hence would be a homeomorphism of the space $\mathrm{Spec}(R)$. As the ordering is an antichain we can map points anywhere, so every singleton would likewise be clopen. This would imply that $\mathrm{Spec}(R)$ is an infinite discrete space, contradicting compactness of the space $\mathrm{Spec}(R)$. Thus the topology can not be recovered from the inclusion ordering.

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