Exercise 7. *(Chriestenson, Tuley, Wane) Show that*

(1) *the only idempotents of a local ring are* 0 *and* 1.
(2) *there are nonlocal rings whose only idempotents are* 0 *and* 1.
(3) *if R is Artinian and its only idempotents are* 0 *and* 1, *then R is local. Conclude that an Artinian ring is a finite product of local rings.*

Proof. To begin let R be a local ring with unique maximal ideal $\mathfrak{m}$. Recall that ever element not in $\mathfrak{m}$ is a unit and if $r \in \mathfrak{m}$ then $1-r$ is a unit. Let $e \in R$ such that $e^2 = e$ then $0 = e^2 - e = e(e-1)$. If $e \in \mathfrak{m}$ then $1-e$ is a unit and hence not a zero divisor. Thus $e = 0$. If $e \notin \mathfrak{m}$ then e is a unit and hence not a zero divisor. Thus $1 - e = 0$ and so $e = 1$.

Let $R = \mathbb{Z}$. All idempotents of R satisfy the equation $x^2 - x = 0$. Thus the only idempotents of R are 0 and 1. Hence the only idempotents of R are 0 and 1. However since every prime $p \in R$ generates a maximal ideal we have that R is not a local ring. Hence nonlocal rings having only 0 and 1 as idempotents do exist.

Let R be an Artinian ring whose only idempotents are 0 and 1. Let $r \in R$ be a non unit element. Then $(r) \supseteq (r)^2 \supseteq (r)^3 \supseteq \cdots$. This is a descending chain of ideals so there exists $k \in \mathbb{Z}_{\geq 0}$ such that for all $l \in \mathbb{Z}_{\geq 0}$ we have that $(r)^k = (r)^{k+l}$. In particular $(r)^k = (r)^{2k} = (r)^k(r)^k$ which implies $I = (r)^k$ is an idempotent ideal. Since I is the finite product of finitely generated ideals it is finitely generated hence it is generated by one idempotent element. Therefore $I = (0)$ or $I = (1) = R$. Since r is a non unit $(r)^k \neq R$, and hence we must have $I = (0)$. Thus every non unit in R generates a nilpotent ideal and hence is nilpotent. Therefore every non unit in R is in $\mathrm{Nil}(R)$, and thus every element not in $\mathrm{Nil}(R)$ is a unit. Therefore R is a local ring with $\mathrm{Nil}(R)$ as its maximal ideal.

Let R be an Artinian ring. For clarity we will state a few facts.

(1) If R is a commutative ring and $e, f \in R$ are two idempotent elements such that $(e) = (f)$ then $e \in (f)$ and $f \in (e)$. Note that for each $r \in (e)$ and $s \in (f)$ we have that $(1-e)s = 0$ and $(1-f)r = 0$. Thus we get $0 = (1-e)f = f - ef$ and $0 = (1-f)e = e - fe$. This gives $e - ef = f - ef$ so $e = f$. Hence elements generating the same idempotent ideal are equal.
(2) If R is Artinian and $r \in R$ is not nil then $(r) \supseteq (r^2) \supseteq \cdots$ does not become zero and eventually becomes constant. Hence there is a $k \geq 0$ such that $(r^k) = (r^{2k}) = (r^k)^2$. Thus the ideal (r^k) is

idempotent. Since it it finitely generated there is an idempotent element $s \in R$ such that $(s) = (r^k)$. Therefore $s = r^k$.

(3) If R is Artinian the set of ideals generated by idempotent elements is a subset of $\mathrm{Ideal}(R)$ and thus has a minimal element, (e).
(4) The element e gives us that $R \cong R/(1-e) \times R/(e)$.
(5) Since $R \cong (e) \oplus (1-e)$ we have that each coset of $(1-e)$ in $R/(1-e)$ can be represented by an element of (e). Hence we may assume elements of $R/(1-e)$ have the form $er + (1-e)$ for $r \in R$. In particular $1 + (1-e) = e + (1-e)$.
(6) If $\pi : R \to R/(1-e)$ is the canonical projection we know that $\pi(\mathrm{Nil}(R)) \subseteq \mathrm{Nil}(R/(1-e))$. Hence if $er + (1-e)$ is not nil in $R/(1-e)$ then er is not nil in R.

Let $er + (1-e)$ be not nilpotent in $R/(1-e)$. Then by above er is not nilpotent in R. Also by above there exists an idempotent element $s \in R$ and an integer $k \geq 0$ such that $s = (er)^k$. This implies that $e|s$ so $(s) \subseteq (e)$. Since (s) is generated by a idempotent element our choice of (e) gives us that either $(s) = (0)$ or $(s) = (e)$. Since $(er)^k \neq 0$ we have that $(s) \neq (0)$. Thus we must have that $((er)^k) = (s) = (e)$. This means $(er)^k = e$, giving us that $er^k = e^k r^k = (er)^k = e$. Hence r^k acts as the identity on e. Thus $r^k + (1-e) = 1 + (1-e)$, and $r + (1-e)$ is a unit in $R/(1-e)$. Hence so is $er + (1-e)$. We have shown that every element of $R/(1-e)$ not in $\mathrm{Nil}(R/(1-e))$ is a unit, hence $R/(1-e)$ is a local ring with maximal ideal $\mathrm{Nil}(R/(1-e))$.

This shows that we have a decomposition of R into $R \cong R_1 \oplus S_1$ where $R_1 \cong R/(1-e)$ is local and $S_1 \cong R/(e)$. Since quotients of Artinian rings are again Artinian we have that S_1 is Artinian. Hence we can repeat this process with S_1 to get $S_1 = R_2 \times S_2$ where R_2 is local. We can continue decomposing R in this manner. Note that for each $i \geq 1$ we have the projection map p_i from R onto $R_1 \times \cdots \times R_i$, and $\mathrm{Ker}(p_1) \supseteq \mathrm{Ker}(p_2) \supseteq \cdots$. Thus for some k large enough we have $\mathrm{Ker}(p_k) = \mathrm{Ker}(p_{k+l})$, for all $l \geq 0$. This implies that $R \cong R_1 \times R_2 \times \cdots \times R_k$ and each R_i is local. Hence R is a finite product of local rings. $\square$