

Problem 1. (Gern Moorhead, Strider). Suppose that $I \triangleleft R$ has infinitely many minimal primes that contain it.

- (a) Show that I is not prime.
- (b) Use (a) to show that there is an ideal properly containing I that also has infinitely many minimal primes above it.
- (c) Conclude that R is not Noetherian. (Expressed more positively, any Noetherian ring has the property that every ideal I has only finitely many minimal primes containing it, hence \sqrt{I} is an intersection of finitely many primes.)

Solution: Let $I \triangleleft R$ such that I is contained in infinitely many primes that are minimal over I .

- (a) If there are infinitely many minimal primes above I , then there are certainly two distinct minimal primes, say P and Q , above I . Assume I is prime. Then, because $I \subseteq P$ and P is minimal with respect to the property of being prime, we must have $I = P$. Similarly, $I = Q$. But P and Q were distinct, so this is a contradiction. Thus I is not prime.
- (b) In part (a) we showed that I is not prime, so there exist $x, y \in R$ such that $xy \in I$ but $x \notin I$ and $y \notin I$. Then we see that $I \subsetneq I + (x)$ and $I \subsetneq I + (y)$. Let P be a prime that is minimal over I . Then we see that $(I + (x))(I + (y)) = I + I \cdot (x) + I \cdot (y) + (xy) \subset I \subset P$, so either $I + (x) \subset P$ or $I + (y) \subset P$. Then since there are infinitely many minimal primes P containing I , we see that either $I + (x)$ or $I + (y)$ must be contained in infinitely many primes that are minimal over I . Since each of these primes is minimal over I , and $I \subset I + (x)$ and $I \subset I + (y)$, then each of these primes is minimal over $I + (x)$ or $I + (y)$, respectively.
- (c) Without loss of generality, suppose that $I_1 = I + (x)$ is contained in infinitely many primes that are minimal over I_1 . By part (a) we see that I_1 is not prime, and by part (b) there exists some ideal I_2 properly containing I_1 that is also contained in infinitely many primes that are minimal over I_2 . We see that we may continue this process as many times as we want, so we can find a chain,

$$I = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

of ideals that is infinite. Therefore, R does not satisfy the ascending chain condition, so R is not Noetherian.

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