COMMUTATIVE ALGEBRA HW 2

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Problem 5

Prove that the intersection of a chain of prime ideals is a prime ideal. Conclude that every ideal I has minimal prime ideals that contain it and that $\sqrt{I}$ is the intersection of these minimal primes.

Solution

Let $\{P_\alpha\}$ be a chain of prime ideals in a ring R, that is to say a collection of prime ideals that is totally ordered by inclusion. Let

$$P = \bigcap_\alpha P_\alpha$$

be the intersection of this chain. Since $0 \in P_\alpha$ for all α we have that $0 \in P$ and hence P is nonempty. Let $x, y \in P$ and let $r \in R$. Then for each α, since P_α is an ideal we have that $x - y \in P_\alpha$ and $rx \in P_\alpha$. Then $x - y \in P$ and $rx \in P$. It follows that P is an ideal. Now let $a, b \in R$ and suppose that $ab \in P$ and $a \notin P$. Then $ab \in P_\alpha$ for all α. Since $a \notin P$, there exists an index β such that $a \notin P_\beta$. Let P_α be an ideal in the chain. Since the chain is totally ordered by inclusion we must have $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$. Suppose $P_\alpha \subseteq P_\beta$. Then since $a \notin P_\beta$ we have $a \notin P_\alpha$. Since P_α is prime and contains ab but not a it follows that $b \in P_\alpha$. Now suppose instead that $P_\beta \subseteq P_\alpha$. Since P_β is prime and contains ab but not a we must have that $b \in P_\beta$. Then $b \in P_\alpha$. Since P_α was arbitrary, it follows that $b \in P_\alpha$ for all α. Then $b \in P$ and it follows that P is prime.

Now let I be an ideal and consider that the set of prime ideals that contain I forms a partially ordered set under reverse inclusion. This set is nonempty since R is such a prime ideal. The above argument implies that every ascending chain under this order has an upper bound, namely the intersection of the prime ideals in the chain. Then by Zorn's lemma there is a maximal element under this ordering. This maximal element is a prime ideal containing I that does not properly contain another prime ideal that contains I, and hence is a minimal prime ideal containing I.

Let I be an ideal and let $\{P_\alpha\}$ be the collection of minimal prime ideals containing I. Let P be the intersection of these minimal prime ideals. Suppose $a \in \sqrt{I}$. Then $a^n \in I$ for some positive integer n. Then a^n in P_α for every α and hence $a^n \in P$. Since P is prime it follows that $a \in P$. Thus $\sqrt{I} \subseteq P$. Now suppose $a \in P$ where $a \neq 0$ (the result is trivial if $P = (0)$). Let $S = \{a, a^2, a^3, \ldots\}$. Since S is multiplicatively closed and does not contain zero, a previous result implies that any ideal J that is maximal such that $J \cap S = \varnothing$ must be prime. Suppose $a \notin \sqrt{I}$. Then no power of a is contained in I and hence $I \cap S = \varnothing$. Then the collection of ideals that are disjoint from S is nonempty, since I is a member. We may therefore take J to be the union of all ideals disjoint from S. Thus defined, J is maximal among ideals that are disjoint from S and hence is a prime ideal containing I. Then J contains a minimal prime ideal that contains I. This minimal prime must be one of the ideals P_α. This minimal ideal does not contain a since J does not contain a. Then $a \notin P$, a contradiction. Then our supposition was false and it follows that $a \in \sqrt{I}$. Then $P \subseteq \sqrt{I}$ and hence $P = \sqrt{I}$ as required. $\square$