Commutative Algebra Assignment 2, Problem 4

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(a) To show that the collection of radical ideals ordered by inclusion is a complete lattice, we need to show that for any arbitrary collection of radical ideals, there exists a least upper bound and greatest lower bound. Let $\{x_i\}$ be a collection of radical ideals, and let I be a radical ideal containing all the x_i. We have that I contains the ideal $\sum x_i$, and so we must have $I = \sqrt{I} \supset \sqrt{\sum x_i}$. Hence, every radical ideal containing all the x_i also contain the radical ideal $\sqrt{\sum x_i}$, so we have that $\sqrt{\sum x_i}$ is a least upper bound in the radical ideal lattice for the collection $\{x_i\}$.

Now, consider $\bigcap x_i$. Suppose $r^n \in \bigcap x_i$ for some ring element r and positive integer n. Then $r^n \in x_i \implies r \in \sqrt{x_i} = x_i$ for all i. Hence, $r \in \bigcap x_i$, so $\bigcap x_i$ is a radical ideal. Indeed, since $\bigcap x_i$ is the largest set contained in all the x_i, it follows that $\bigcap x_i$ is the least upper bound in the ideal lattice for the collection $\{x_i\}$.

(b) Let y be a radical ideal, and $\{x_i\}$ a family of radical ideals. The complete distributive law takes the following form in the lattice of radical ideals:

$y \cap \sqrt{\sum x_i} = \sqrt{\sum y \cap x_i}$

Let $r \in y \cap \sqrt{\sum x_i}$. Then $r \in \sqrt{\sum x_i}$. By definition, this means that there exists a finite collection of $z_i \in x_i$ and a natural number n such that $r^n = \sum z_i$. Multiplying both sides by r gives $r^{n+1} = \sum r z_i$. Now since $r \in y$ and $z_i \in x_i$, we have $r z_i \in y \cap x_i$ for each i. Thus $r^{n+1} \in \sum y \cap x_i$, so we have $r \in \sqrt{\sum y \cap x_i}$. Thus $y \cap \sqrt{\sum x_i} \subseteq \sqrt{\sum y \cap x_i}$.

Let $r \in \sqrt{\sum y \cap x_i}$. Then there exists a finite collection of $z_i \in y \cap x_i$ and a natural number n such that $r^n = \sum z_i$. Since each $z_i \in y$, $r^n \in y$. y is a radical ideal, so $r \in y$. Moreover, $r^n = \sum z_i \in \sum x_i$, so $r \in \sqrt{\sum x_i}$. We thus have $r \in y \cap \sqrt{\sum x_i}$, proving that $y \cap \sqrt{\sum x_i} \supseteq \sqrt{\sum y \cap x_i}$. The complete distributive law follows.

(c) The map $\sqrt{\ }: I \mapsto \sqrt{I}$ is clearly a surjective map from Ideal(R) to the lattice of radical ideals. Let a_i be a collection of ideals. Let $\vee$ denote the join in the lattice of radical ideals. To show that $\sqrt{\ }$ is a $+$-complete lattice homomorphism, we need to show that $\bigvee \sqrt{a_i} = \sqrt{\sum a_i}$ and $\sqrt{a} \cap \sqrt{b} = \sqrt{a \cap b}$.

From part a, $\bigvee \sqrt{a_i} = \sqrt{\sum \sqrt{a_i}}$. Since for each i, $a_i \subseteq \sqrt{a_i}$, we have that $\sum a_i \subseteq \sum \sqrt{a_i}$, and so $\sqrt{\sum a_i} \subseteq \sqrt{\sum \sqrt{a_i}} = \bigvee \sqrt{a}_i$. To see the reverse inclusion, note that $\bigvee \sqrt{a}_i$ is the smallest radical ideal containing each $\sqrt{a}_i$. Since $a_i \subseteq \sum a_i$ for all i, we have that $\sqrt{a}_i \subseteq \sqrt{\sum a_i}$ for each i. But $\sqrt{\sum a_i}$ is a radical ideal, so $\bigvee \sqrt{a}_i \subseteq \sqrt{\sum a_i}$. Thus $\bigvee \sqrt{a}_i = \sqrt{\sum a_i}$, so $\sqrt{\ }$ preserves the complete join operation.

Finally, let a, b be two ideals in R. Recall from part a that the meet of two radical ideals is simply their intersection. Since $a \cap b \subset \sqrt{a} \cap \sqrt{b}$, $\sqrt{a \cap b} \subset \sqrt{a} \cap \sqrt{b}$. Conversely, if $r \in \sqrt{a} \cap \sqrt{b}$, then there exists a positive integer n such that $r^n \in \sqrt{a} \cap \sqrt{b}$. That is, $r^n \in \sqrt{a}$ and $r^n \in \sqrt{b}$. Hence, there exist positive integers m_1, m_2 such that $r^{nm_1} \in a$ and $r^{nm_2} \in b$. Hence, $r^{nm_1m_2} \in a \cap b \implies r \in \sqrt{a \cap b}$. Hence, $\sqrt{a} \cap \sqrt{b} \subset \sqrt{a \cap b}$ and so $\sqrt{a} \cap \sqrt{b} = \sqrt{a \cap b}$.