**Problem 3** (Pratarelli, Selker). *(a)* $\operatorname{rad}(R)$ *is the largest ideal* $J \lhd R$ *such that all covers below* $J$ *in Ideal*$(R)$ *are of abelian type. (That is,* $I \prec K \leq J$ *implies* $K^2 \subseteq I$*.)*

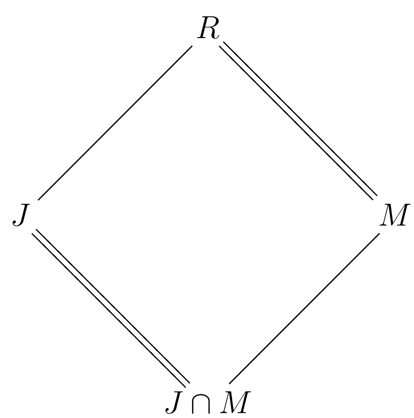
*(b)* $\operatorname{nil}(R)$ *is the largest ideal* $I \lhd R$ *such that there is a well ordered chain* $0 = I_0 \leq I_1 \leq \ldots \leq_\mu = I$ *such that*

*(i)* $I_{\alpha+1}$ *is abelian over* $I_\alpha$ *for all* $\alpha$*, and*

*(ii)* $I_\lambda = \bigcup_{\kappa<\lambda}$ *whenever* $\lambda$ *is a limit ordinal.*

*Proof.* (a) First we show that if $J$ is the Jacobsen radical and $I \prec K \leq J$ then $K$ is abelian over $I$. By minimality of the cover $I \prec K$, we have that $K/I$ is a simple $(R/I)$-module. By problem 5 on assignment 1, we have that $J/I \subseteq \operatorname{rad}(R/I)$, thus $K/I \subseteq \operatorname{rad}(R/I)$, so $K/I$ annihilates the $(R/I)$-module $K/I$, i.e. $K^2/I = 0/I$, so $K^2 \subseteq I$, as needed.

Now suppose that $J \not\subseteq \operatorname{rad}(R)$. Thus there is a maximal ideal $M$ such that $M \not\supseteq J$, hence $M + J = R$ and $[M, R]$ and $[M \cap J, J]$ are perspective intervals of the modular lattice Ideal(R), hence are isomorphic.




So $(J \cap M) \prec J$ and the cover is not abelian because the perspective cover $M \prec R$ is not abelian ($R^2 \not\subseteq M$).

(b) First we claim that any such chain of ideals is contained in $\operatorname{nil}(R)$, i.e. if $0 = I_0 \leq I_1 \leq \ldots \leq I_\mu = I$, such that conditions (i) and (ii) above hold, then $I \subseteq \operatorname{nil}(R)$. To see this, suppose that $i \in I = I_\lambda$. Let $\alpha_n$ be minimal such that $i^{2^n} \in I_{\alpha_n}$. Then $\alpha_0 > \alpha_1 > \ldots$ is a descending chain of ordinals, thus it must have finite length, so for some $n \in \omega$, $\alpha_n = \alpha_{n+1}$. It follows by (i), (ii), and the choice of $\alpha_{n+1}$ that $\alpha_n = 0$, thus $i^{2^n} = 0$, so $i \in \operatorname{nil}(R)$.

Thus we have shown that if $I$ is the terminus of any chain as in (i) and (ii), then $I \subseteq \operatorname{nil}(R)$, so it suffices to build such a chain whose terminus contains $\operatorname{nil}(R)$.

We claim:

$$\text{If } J \lhd R \text{ and } \forall \bar{r} \in (R/J) \setminus \{\bar{0}\}, \bar{r}^2 \neq 0 \text{ then } \operatorname{nil}(R) \subseteq J. \tag{1}$$

Suppose that $\mathrm{nil}(R) \not\subseteq J$, then let $\bar{r} \in \mathrm{nil}(R) \setminus J$. Then $\bar{r} \neq \bar{0}$, but $\bar{r}^{(2^n)} = 0$ for some $n$; assume that this $n$ is minimal. Then $\bar{s} := \bar{r}^{(2^{n-1})} \neq 0$ but $\bar{s}^2 = 0$, so (1) is proved by contraposition.

Now we define a chain of ideals by recursion. Let $I_0 = (0)$. Suppose that $I_\alpha$ has been defined for an ordinal $\alpha$. If there is a nonzero element $\bar{r}_\alpha \in R/I_\alpha$ such that $\bar{r}_\alpha^2 = \bar{0}$ then let $I_{\alpha+1} = I_\alpha + (r_\alpha)$, where $r_\alpha$ is any representative of $\bar{r}_\alpha$. If no such $\bar{r}_\alpha$ exists, then by (1) $I_\alpha \supseteq \mathrm{nil}(R)$, and the construction is complete. If $\lambda$ is a limit ordinal and $I_\alpha$ has been defined for every $\alpha < \lambda$ then let $I_\lambda = \bigcup_{\alpha<\lambda}$. By construction and (1), if $I$ is the terminus of this chain of ideals, $\mathrm{nil}(R) \subseteq I$, so we merely need to see that the construction satisfies conditions (i) and (ii).

We prove (i) by induction on $\alpha$. If $\alpha = 0$ then $I_1$ is $(i)$ where $i^2 = 0$, so $I_1^2 = (i^2) = (0)$, as needed. Suppose that (i) holds for all $\beta \leq \alpha$. Now $I_{\alpha+1} = I_\alpha + (r_\alpha)$ where $r_\alpha^2 \in I_\alpha$, hence $I_{\alpha+1}^2$ is generated by $\{r_\alpha^2\} \cup \{r_\alpha \cdot i : i \in I_\alpha\} \cup \{i \cdot j : i, j \in I_\alpha\}$. Now the first set is a subset of $I_\alpha$ by construction, while the second two sets are subsets of $I_\alpha$ because $I_\alpha$ is an ideal. $I_{\alpha+1}^2 \subseteq I_\alpha$, completing the induction.

Property (ii) follows directly from the definition of the chain.

$\square$