COMMUTATIVE ALGEBRA: HOMEWORK 2

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2) Prove that the Jacobson radical contains no nonzero idempotents in each of the following ways:

(a) using the characterization of $\mathrm{rad}(R)$ as the intersection of maximal ideals.

(b) using the characterization of $\mathrm{rad}(R)$ as the largest ideal J such that $1 + J$ consists of units.

(c) using the characterization of $\mathrm{rad}(R)$ as the intersection of annihilators of all simple modules.

Solution

(a) *Proof.* Since $\mathrm{rad}(R)$ is the intersection of maximal ideals, it is sufficient to show that if $e \in \mathrm{rad}(R)$ is a nonzero idempotent, then there is maximal ideal $M \trianglelefteq R$ that does not contain e. Let $\mathcal{S}$ be the set of proper ideals of R containing the element $1 - e$ and partially ordered under inclusion. Since $e \neq 0$, $1 - e$ is not a unit (see (b) below) and so $(1-e) \in \mathcal{S}$. Thus $\mathcal{S}$ is not empty. Let $\mathcal{N} = (N_i)_{i \in \mathbb{Z}_{>0}}$ be an ascending chain of ideals in $\mathcal{S}$ and let $N = \bigcup \mathcal{N}$. Since the N_i are nested, N is an ideal and contains $1 - e$, but it could be that N is nonproper. Suppose that this is the case, then $1 \in N$, so there is k such that $1 \in N_k$, which is a contradiction. Hence N is a proper ideal of R and thus an element of $\mathcal{S}$. Therefore each ascending chain has an upper bound, so by Zorn's Lemma there is a maximal element of $\mathcal{S}$, call it M. Suppose that $e \in M$. Then $1 - e + e = 1 \in M$ as well, contradicting M being a proper ideal. $\square$

(b) *Proof.* Let $e \in \mathrm{rad}(R)$ be an idempotent. Then $1 - e \in 1 + \mathrm{rad}(R)$ is a unit. However,

$$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e,$$

so $1 - e$ is also an idempotent. The only idempotent that is also a unit is 1, so it must be that $e = 0$. Therefore there are no nonzero idempotents in $\mathrm{rad}(R)$. $\square$

(c) *Proof.* Let $e \in \mathrm{rad}(R)$ be a nonzero idempotent and let $M \trianglelefteq R$ be a maximal ideal containing $1 - e$ (as in part (a) above). Then R/M is a field and thus a simple R-module. Therefore $\mathrm{rad}(R)$ annihilates R/M, so in particular

$$0 + M = e \cdot (1 + M) = e + M.$$

Therefore $e \in M$, which is a contradiction as in (a) above: $1 - e + e = 1 \in M$, but M is proper. $\square$

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