Math 6150 Commutative Algebra Fall 2009

Assignment 2 Problem 1 **Hill, Jones**

(a) Show that $\text{nil}(R \times S) = \text{nil}(R) \times \text{nil}(S)$ and $\text{rad}(R \times S) = \text{rad}(R) \times \text{rad}(S)$. Hence, if the nilradical and the Jacobson radical are equal in each coordinate of a product, then they are equal in the product.

Proof: Let $(a, b) \in \text{nil}(R \times S)$. Then for some $n \in \mathbb{Z}_{\geq 1}$ we have $(a, b)^n = (a^n, b^n) = (0_R, 0_S)$ in $R \times S$, giving that $a \in \text{nil}(R)$ and $b \in \text{nil}(S)$. Conversely, let $a \in R$ and $b \in S$. If there is some $n, m \in \mathbb{Z}_{\geq 1}$ such that $a^n = 0_R$ and $b^m = 0_S$, then setting $k = \max(m, n)$ gives

$$(a, b)^k = (a^k, b^k) = (a^{n+(k-n)}, b^{m+(k-m)}) = (a^n a^{k-n}, b^m b^{k-m}) = (0_R, 0_S)$$

and so $(a, b) \in \text{nil}(R \times S)$. Next, recall that the maximal ideals of $R \times S$ have the form $R \times M_S$ or $M_R \times S$, where M_R and M_S are maximal ideals of R and S, respectively. Thus, $(c, d) \in \text{rad}(R \times S)$ if and only if $c \in M_R$ for all maximal ideals $M_R \lhd R$ and $d \in M_S$ for all maximal ideals $M_S \lhd S$, if and only if $c \in \text{rad}(R)$ and $d \in \text{rad}(S)$. $\square$

(b) Show the result for part (a) does not hold for infinite products by showing that the nilradical and Jacobson radical are equal in all coordinates of the product $T = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$, but $\text{nil}(T) \neq \text{rad}(T)$.

Proof: Let $R = \mathbb{Z}_{2^n}$ for some $n \in \mathbb{Z}_{\geq 1}$. Let $a \in R \cap (\overline{2})$. Then $a = \overline{2}r$ for some $r \in R$ and we have $a^n = \overline{2}^n r^n = 0 \cdot r^n = 0_R$, and so $(\overline{2}) \subseteq \text{nil}(R)$. Since for all $m \in \mathbb{Z}$ we have the element $\overline{2m+1} \in U(R)$[1], it follows that $\text{nil}(R) = (\overline{2})$. Also, note that $(\overline{2})$ is maximal in R while $(\overline{2}) + 1 \subseteq U(R)$, giving $\text{rad}(R) = \text{nil}(R)$.

Since $\text{rad}(T)$ is the largest ideal such that $\text{rad}(T) + 1_T \subseteq U(R)$, we see immediately from the pointwise operation structure on T that

$$\text{rad}(T) = \prod_{i=1}^{\infty} (\overline{2}) = (\overline{2}) \times (\overline{2}) \times (\overline{2}) \times \cdots \lhd \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$$

since, as before, $(\overline{2})$ is maximal and $(\overline{2}) + \overline{1}$ is a collection of units in each component. However, we claim that $\text{rad}(T)$ strictly contains $\text{nil}(T)$. To see this, note that the nilpotence class of a nilpotent element r (the smallest positive integer n such that $r^n = 0$) is finite. With the given ordering on components of T, for any finite $n \in \mathbb{Z}_{\geq 1}$, we may find a first component in the product (and hence infinitely many thereafter) such that $(\overline{2})^n \neq 0$. Thus, $\text{nil}(T) \neq \text{rad}(T)$. $\square$

[1] Recall that $U(\mathbb{Z}_{2^n})$ consists of the elements of $\mathbb{Z}_{2^n}$ (more precisely, classes of integers modulo n^2) relatively prime to 2^n.