Commutative Algebra Assignment 1 Problem 9

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September 11, 2009

Abstract

Show that if R is a commutative ring, then R is a homomorphic image of a subring of a field. Conclude that commutative rings satisfy all positive universal sentences true in all fields. Explain how this shows that (for example) the truth of the Cayley-Hamilton Theorem for fields implies the truth of this theorem for any commutative ring.

Solution

Let R be a commutative ring. Form the polynomial ring $\mathbb{Z}[X]$, where $X = \{x_r : r \in R\}$ is a set of commuting indeterminates, and each x_r corresponds to one $r \in R$. The degree of a nonzero monomial $a \prod_{\alpha \in \lambda} x_\alpha^{e_\alpha}$ in $\mathbb{Z}[X]$, where $\lambda \subset R$ is finite, is defined to be $\sum_{\alpha \in \lambda} e_\alpha$. We define the degree of a nonzero polynomial f in $\mathbb{Z}[X]$ to be the maximum of the degrees of the monomials making up f. Then for all nonzero $f, g \in \mathbb{Z}[X]$, $deg(fg) = deg(f) + deg(g)$. The degree of the zero polynomial is defined to be $-\infty$. Hence, if f and g are nonzero polynomials, fg is nonzero, so $\mathbb{Z}[X]$ is an integral domain. Thus, $\mathbb{Z}[X]$ is a subring of its field of fractions $\mathbb{F}$.

Consider a map $\phi : \mathbb{Z}[X] \to R$ defined by $\phi(a) = a$ for $a \in \mathbb{Z}$ and $\phi(x_r) = r$ for all $r \in R$ and extended by linearity. Then ϕ is a homomorphism by definition, and ϕ is surjective, since $\phi(x_r) = r$ for all $r \in R$. Thus, R is the homomorphic image of a subring of the field $\mathbb{F}$.

Let s be a positive universal sentence true in all fields. Now, $R = \phi(\mathbb{Z}[X])$, where ϕ is the homomorphism given above. Since s is a universal sentence true in $\mathbb{F}$, it is also true in the subring $\mathbb{Z}[X]$ of $\mathbb{F}$ (see Cor. 2.4.2 in [1]). Further, since s is a positive sentence true in $\mathbb{Z}[X]$, it is also true for the homomorphic image $R = \phi(\mathbb{Z}[X])$ (see Thm. 2.4.3 (b) in [1]). Thus, s is true in R.

Recall the Cayley-Hamilton theorem: any $n \times n$ matrix M over a field $\mathbb{F}$ is a zero of its own characteristic polynomial $c(x)$. We claim this statement may be written as a positive universal sentence in $\mathbb{F}$. Given the characteristic polynomial $c(x)$, the Cayley-Hamilton theorem is equivalent to the matrix equation $c(M) = 0$. This matrix equation is equivalent to n^2 equations over the field $\mathbb{F}$, all of which must be true for all choices of entries for M. Hence, the Cayley-Hamilton Theorem is a positive universal sentence, true in all fields $\mathbb{F}$. Thus, it must be true in commutative rings, as well.

We demonstrate the case $n = 2$. Let M be a matrix over $\mathbb{F}$, with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $c(x) = x^2 - (a+d)x + ad - bc$, so $c(M) = 0$ can be written in the following way:

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= c(M) = M^2 - (a+d)M + (ad-bc)I \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We write this matrix equation as four equations over $\mathbb{F}$:

- $E_1 : 0 = a^2 + bc - (a+d)a + (ad - bc)$
- $E_2 : 0 = ab + bd - (a+d)b$
- $E_3 : 0 = ac + cd - (a+d)c$
- $E_4 : 0 = d^2 + bc - (a+d)d + (ad - bc)$

Thus, the Cayley Hamilton Theorem for 2×2 matrices can be written in the following form: $\forall a, b, c, d(E_1 \& E_2 \& E_3 \& E_4)$. Hence, the Cayley-Hamilton theorem is a positive universal sentence true in all fields $\mathbb{F}$ in the 2×2 case.

References

1. Hodges, Wilfrid. *A Shorter Model Theory.* Cambridge University Press, 1997.