Problem 8 (Selker, Wane)**.** *If R is a commutative ring then* $\operatorname{rad}(R)$ *is the largest ideal J such that $1 + J$ consists of units.*

Proof. We claim first that if $J = \operatorname{rad}(R)$ then $1 + J$ consists of units. Suppose not, then we must have some $j \in J$ with $1 + j$ not a unit. By corollary 1.5 there exists a maximal ideal $\mathfrak{m}$ such that $1 + j \in \mathfrak{m}$. Now note that we have $j \in J \subseteq \mathfrak{m}$, so also $j \in \mathfrak{m}$, but then $(1 + j) - j = 1 \in \mathfrak{m}$, a contradiction.

Now we claim that if $\mathfrak{a} \lhd R$ and $\mathfrak{a} \not\subseteq \operatorname{rad}(R)$ then $1 + \mathfrak{a}$ does not consist (solely) of units. If $a \in \mathfrak{a} \setminus \operatorname{rad}(R)$ then by definition there is some maximal ideal $\mathfrak{m}$ such that $a \notin \mathfrak{m}$. Thus we must have that $\mathfrak{m} + (a) = (1)$, the unit ideal. Thus we have $1 = m + ra$ for some $m \in \mathfrak{m}$ and some $r \in R$. Then $1 - ra = m \in \mathfrak{m}$, so $1 - ra = 1 + (-r)a$ cannot be a unit, and clearly $(-r)a \in \mathfrak{a}$, so the claim is proved. $\square$