COMMUTATIVE ALGEBRA: HOMEWORK 1

MATTHEW MOORE AND ANDREW MOORHEAD

6) Suppose that $I \trianglelefteq R$ is nil.

(a) Show that $a+I$ is a unit in R/I if and only if a is a unit in R.

(b) Show that $a+I$ is idempotent in R/I if and only if there is idempotent $e \in R$ such that $e+I = a+I$.

Solution

(a) *Proof.* Suppose that $a+I \in R/I$ is a unit. Then there is $b+I \in R/I$ such that $(a+I)(b+I) = ab+I = 1+I$. Therefore $ab-1 \in I$. Since I is nil, there is some $n \in \mathbb{Z}_{>0}$ such that $(1-ab)^n = 0$. Using the binomial expansion, we have

$$0 = (1-ab)^n = \sum_{i=0}^{n} \binom{n}{i}(-ab)^i(1)^{n-i} = 1 + \sum_{i=1}^{n}\binom{n}{i}(-1)^i a^i b^i$$
$$= 1 + a\sum_{i=1}^{n}\binom{n}{i}(-1)^i a^{i-1}b^i.$$

Therefore $a\left(-\sum_{i=1}^{n}\binom{n}{i}(-1)^i a^{i-1}b^i\right) = 1$, so a is a unit in R.

Suppose that $a \in R$ is a unit. Then there is $b \in R$ such that $ab = 1$. Let $\pi : R \to R/I$ be the natural homomorphism. Then

$$1 + I = \pi(1) = \pi(ab) = ab + I = (a+I)(b+I).$$

Thus $a+I$ is a unit in R/I. $\square$

(b) Solution 1

Proof. Suppose that $a+I$ is idempotent in R/I. Then $a+I = a^2+I$, so $a(1-a) \in I$. Since I consists of nilpotent elements, there is $n \in \mathbb{Z}_{>0}$ such that $a^n(1-a)^n = 0$. Furthermore, $a+I$ idempotent implies that $(1-a)+I$ is idempotent. It follows from this that

$$a^n + (1-a)^n + I = a + (1-a) + I = 1 + I.$$

Hence $a^n + (1-a)^n + I$ is a unit in R/I, so $a^n + (1-a)^n$ is a unit in R by part (a) above. Let $u = [a^n + (1-a)^n]^{-1}$ and let $e = ua^n$. We have

$$e = e(1) = (ua^n)\left(u[a^n + (1-a)^n]\right) = u^2a^{2n} + u^2a^n(1-a)^n = u^2a^{2n} = (ua^n)^2$$
$$= e^2.$$

Therefore $e = ua^n$ is idempotent in R. Since $a^n + (1-a)^n + I = 1 + I$, it must be that $u + I = 1 + I$. Therefore

$$e + I = ua^n + I = (u+I)(a^n+I) = a + I.$$

Date: September 4, 2009.

Suppose that $e \in R$ is idempotent and let $\pi : R \to R/I$ be the natural homomorphism. Then $e + I = \pi(e) = \pi(e^2) = e^2 + I = (e + I)^2$, so $e + I$ is idempotent in R/I. □

Solution 2

Proof. Suppose that $a+I$ is idempotent in R/I. Then $a+I = a^2+I$, so $a(1-a) \in I$. Since I consists of nilpotent elements, there is $n \in \mathbb{Z}_{>0}$ such that $a^n(1-a)^n = 0$. Using the binomial expansion, we have

$$1 = (a + (1-a))^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} (a)^i (1-a)^{2n-i}$$

Noting that the *nth* term of this series is zero, we let

$$e = 1 - \sum_{i=0}^{n-1} \binom{2n}{i} (a)^i (1-a)^{2n-i} = \sum_{i=n+1}^{2n} \binom{n}{i} (a)^i (1-a)^{2n-i}$$

Now, as $a(1-a) \in I$, we have that $(a)^i(1-a)^{2n-i} \in I$ for $n+1 \leq i < 2n$. Therefore, $e + I = a^n + I = a + I$. Furthermore, e is idempotent, as

$$\begin{aligned} e^2 &= \left(\sum_{i=n+1}^{2n} \binom{n}{i} (a)^i (1-a)^{2n-i} \right) \left(1 - \sum_{i=0}^{n-1} \binom{2n}{i} (a)^i (1-a)^{2n-i} \right) \\ &= e + 0 \end{aligned}$$

□