Assignment I

Commutative Algebra, Math 6150
Kearnes 12:00

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Problem #5

Proposition. *Let $\phi : R \to S$ be a surjective homomorphism of commutative rings. Then we have that $\phi(\mathrm{nil}(R)) \subseteq \mathrm{nil}(S)$ and $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$. Under the additional hypothesis that $\ker\phi$ is nil, we find $\phi(\mathrm{rad}(R)) = \mathrm{rad}(S)$. Further, the hypothesis $\ker\phi$ is nil is not trivial.*

Proof.
First we show that $\phi(\mathrm{nil}(R)) \subseteq \mathrm{nil}(S)$.

Suppose that $s \in \phi(\mathrm{nil}(R))$. Then there exists an $r \in R$ such that $\phi(r) = s$ and an $n \in \mathbb{Z}^+$ such that $r^n = 0$. Because ϕ is a ring homomorphism, it preserves multiplication, so that $\phi(r^n) = \phi(r)^n$. Also we know that $\phi(r^n) = \phi(0) = 0$ by an elementary property of ring homomorphisms. Thus

$$s^n = \phi(r)^n = \phi(r^n) = 0,$$

that is, s is nilpotent and therefore in $\mathrm{nil}(S)$. Because s was arbitrarily chosen, our desired result, $\phi(\mathrm{nil}(R)) \subseteq \mathrm{nil}(S)$, follows.

Next we show that $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$.

One particularly clear way to show this result is to recall from general algebra that, for commutative ring R, we know that $a \in \mathrm{rad}(R)$ if and only if $1 - ab$ is a unit for all $b \in R$. (Briefly, we can see this by looking at the ideal I generated by $1 - ab$. If $1 - ab$ is not a unit for a given b, then I is proper and thus in some maximal ideal. But a is in all maximal ideals, so $a \in I$, implying $ab \in I$. Therefor $1 - ab + ab = 1$ is in I, contradicting that it is proper.)

Now suppose that $y \in \phi(\mathrm{rad}(R))$. We will show that $1 - ys$ is a unit for all $s \in S$, implying by the above that $y \in \mathrm{rad}(S)$. Because ϕ is surjective, we know there are $x, r \in R$ such that $\phi(x) = y$ and $\phi(r) = s$. In fact, we can assume $x \in \mathrm{rad}(R)$ because $y \in \phi(\mathrm{rad}(R))$. By the characterization above, we know that $1 - xr$ is a unit in R.

By the properties of homomorphisms, we know that

$$\phi(1 - xr) = \phi(1) - \phi(x)\phi(r) = (1 - ys).$$

Since homomorphisms also preserve units—the inverse of $(1-ys)$ is just the image of the inverse of $(1-xr)$—we have that $(1 - ys)$ is a unit as needed. Since s was chosen arbitrarily, we have that $y \in \mathrm{rad}(S)$. Thus $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$.

We also offer the following alternative proof because, while longer and less elegant, it uses a more ideal-based approach which may elucidate other aspects of the result. Recall from the first isomorphism theorem that $R/\ker\phi$ is isomorphic to $\phi(R) = S$, since ϕ is surjective. In particular, the ideals of $R/\ker\phi$ are in one-to-one correspondence with those from S. Let $\tilde{\phi} : R/\ker\phi \to S$ be this correspondence, which is defined by $\tilde{\phi}(r + \ker\phi) = \phi(r)$.

Now let M be a maximal ideal of S. This corresponds to a maximal ideal of $R/\ker\phi$. Recall that the forth isomorphism theorem states that there is a one-to-one correspondence between ideals of $R/\ker\phi$ and ideals of R that contain $\ker\phi$, so there is an ideal N of R that corresponds to this maximal ideal of $R/\ker\phi$. (Also from the forth isomorphism theorem, we know that this maximal ideal is simply $N/\ker\phi$.) Further, the ideal N is maximal among ideals containing $\ker\phi$. Of course, any ideal containing N clearly contains $\ker\phi$ as well, so N is maximal among all ideals of R.

We can quickly see that N is, in fact, the preimage of M under ϕ. Let $n \in N$ and recall that $\phi(n)$ is equal to $\tilde{\phi}(n + \ker\phi)$. Since $n + \ker\phi \in N/\ker\phi$ and $N/\ker\phi$ corresponds to M, we get that $\tilde{\phi}(n + \ker\phi)$ is in M. Thus $\phi(N) \subseteq M$. On the other hand, if $r \in R \setminus N$, then $r + \ker\phi$ is not in $N/\ker\phi$, so its image under $\tilde{\phi}$ is not in M. Therefore $\phi^{-1}(M) \subseteq N$, so they must be equal.

We have shown that the preimage of a maximal ideal is also maximal for a surjective ring homomorphism. Now, let $r \in \mathrm{rad}(R)$. By definition, we know r is in every maximal ideal of R. We claim that $\phi(r)$ must then be in every maximal ideal of S and thus in $\mathrm{rad}(S)$ as needed.

Suppose there is a maximal ideal M of S that does not contain $\phi(r)$. Then $\phi^{-1}(M)$ must not contain r. However, as shown above, $\phi^{-1}(M)$ is maximal, so it must contain r, a contradiction, completing our second proof that $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$.

Only a few more observations are necessary to show that, when $\ker\phi$ is nil, we have $\phi(\mathrm{rad}(R)) = \mathrm{rad}(S)$. Recall from algebra that, in a commutative ring with unity, all maximal ideals are prime ideals and must then contain all nilpotent elements as follows.

Let P be any prime ideal in R and let a be a nilpotent element of R with $a^n = 0$. Because $0 \in P$, the primality of P tells us that either $a \in P$ or $a^{n-1} \in P$, since their product is. If the former, we are done. If the latter, then we again use the primality of P to get that either $a \in P$ or $a^{n-2} \in P$. Because n is a positive integer, we eventually must have $a \in P$ as needed.

So we have that every nilpotent element of R is in every prime ideal of R and, since all maximal ideals are prime, in every maximal ideal of R. Now, because $\ker\phi$ is nil, i.e. made up entirely of nilpotent elements, it immediately follows that every maximal ideal of R contains the kernel of ϕ. This is convenient because, in the proof above that $\phi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$, we found that the maximal ideals of S correspond directly to the maximal ideals of R containing the kernel of ϕ. So, if we have an element $s \in \mathrm{rad}(S)$, its preimage is contained in every maximal ideal of R containing the kernel, which is now every maximal ideal. Thus it's preimage is in $\mathrm{rad}(R)$, so $s \in \phi(\mathrm{rad}(R))$ as needed to establish equality.

Finally, we show that the requirement that $\ker\phi$ is nil is not a trivial one, i.e. that it is sometimes the case that $\phi(\mathrm{rad}(R)) \subsetneqq \mathrm{rad}(S)$.

Let $R = \mathbb{Z}$ and $S = \mathbb{Z}/8\mathbb{Z}$. Let $\phi : R \to S$ be the standard map sending an integer a to $a \pmod 8$. This is a well-known ring homomorphism and clearly surjective. We observe that the kernel of ϕ is all multiples of 8. Because 0 is the only nilpotent element of $\mathbb{Z}$, it is not the case the $\ker\phi$ is nil, so this is an example of a situation not satisfying the hypothesis above. We will see that, indeed, $\phi(\mathrm{rad}(R)) \subsetneqq \mathrm{rad}(S)$.

It is well-known that the maximal ideals of the integers are just the prime ideals, i.e. $p\mathbb{Z}$ where p is prime. As the primes are infinite, there is no nonzero integer divisible by every prime, so the intersection of all these ideals is 0. That is, we have $\mathrm{rad}(R) = (0)$. Clearly then, $\phi(\mathrm{rad}(R))$ is also zero.

The radical of S is a more complicated object. All proper ideals of S can contain no odd numbers since they are all units in $\mathbb{Z}/8\mathbb{Z}$. On the other hand, the ideal $2\mathbb{Z}/8\mathbb{Z}$ is proper and consists of all the even numbers, so it must be maximal, as if it contained any more, it would contain an odd number and thus a unit. Further, all proper ideals of S must be made up entirely of even number and must therefore be contained in $2\mathbb{Z}/8\mathbb{Z}$, so this is the only maximal ideal of S. Therefore $\mathrm{rad}(S) = 2\mathbb{Z}/8\mathbb{Z}$, which is clearly not (0). Thus $\phi(\mathrm{rad}(R))$ is properly contained in $\mathrm{rad}(S)$ as needed, and the proof is complete. $\square$