Math 6150, Homework 1

Adam Lizzi and Yingwei Li

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#4. Show the ideals of $R \times S$ are of the form $I \times J$ where $I \lhd R$ and $J \lhd S$. Show that the prime (maximal) ideals have the form $P \times S$ or $R \times Q$, where P and Q are prime (maximal) ideals of R and S respectively.

Proof. Anything of the form $I \times J$ is an ideal of $R \times S$ as multiplication in the product is coordinatewise. Now take M to be an ideal of $R \times S$. Let $I = \{i \in R : (i, 0) \in M\}$. Similarly, let $J = \{j \in S : (0, j) \in M\}$. The ideal structure of M implies that I and J are ideals of R and S respectively. (In painstaking detail: if $i_1, i_2 \in I$, then $(i_1, 0), (i_2, 0) \in M$, and as M is an ideal $(i_1, 0) + (i_2, 0) = (i_1 + i_2, 0) \in M$, implying $i_1 + i_2 \in I$. Similarly, I is closed under multiplication by elements in R: if $i \in I$ and $r \in R$, then $(i, 0) \in M$, and for any $s \in S$ we have $(i, 0)(r, s) \in M$, but this product is $(ir, 0)$, implying $ir \in I$ as needed. Symmetric arguments work for J.)

We claim $M = I \times J$. To see that $M \subseteq I \times J$, take $m \in M$, say $m = (x, y)$; then $(1, 0)m = (x, 0) \in M$, and similarly $(0, 1)m = (0, y) \in M$. This implies $x \in I$ and $y \in J$, so that $m \in I \times J$. In the other direction, take $x \in I$ and $y \in J$. We hope $(x, y) \in M$. Since $x \in I$, we have $(x, 0) \in M$. Similarly $(0, y) \in M$, and thus the sum $(x, 0) + (0, y) = (x, y) \in M$ as desired.

Now let $\mathfrak{p}$ be a prime ideal of $R \times S$. By what we've done so far $\mathfrak{p} = P \times Q$ for some ideal $P \lhd R$ and $Q \lhd S$. The calculation $(P \times S)(R \times Q) = (P \times Q)$ shows that at most one of the two factors can be proper. (Since $\mathfrak{p}$ is prime, one of $(P \times S)$ or $(R \times Q)$ must be contained in $\mathfrak{p}$, which is absurd if both factors are proper.) The proper factor must be prime itself. Without loss of generality let $\mathfrak{p} = P \times S$. Then if P is not prime there exist $x, y \in R \backslash P$ with $xy \in P$. But then $(x, 0)$ and $(y, 0)$ are not in $\mathfrak{p}$ and their product $(xy, 0)$ is, contradicting primality of $\mathfrak{p}$.

If $\mathfrak{m}$ is a maximal ideal of $R \times S$, then $\mathfrak{m}$ is prime. Then $\mathfrak{m} = I \times J$ and at most one of I and J is proper. Without loss of generality let $\mathfrak{m} = I \times S$. I must be maximal. If it weren't, say there existed a I' with $I \subset I' \subset R$, then we'd have $(I \times S) \subset (I' \times S) \subset (R \times S)$, contradicting maximality of $\mathfrak{m}$. $\square$