

COMMUTATIVE ALGEBRA HW 1

JONES, KELLER

3. (a) Show that a ring R is directly decomposable as a ring iff it is directly decomposable when considered as an R -module.
- (b) Show that an R -module M is directly decomposable iff it has an idempotent endomorphism $\varepsilon: M \rightarrow M$ such that $\ker(\varepsilon) \neq 0 \neq \text{im}(\varepsilon)$.
- (c) Show that the R -module endomorphisms of ${}_R R$ all have the form $\varepsilon(x) = r(x)$ for some $r \in R$.
- (d) Show that any direct decomposition of R has the form $R \cong R/(e) \times R/(1-e)$ for some idempotent $e \in R$.

Solution (a) Let R be a commutative ring with unit. Then we can consider the R -algebra

$$A = \langle R; +, -, 0, \{\varepsilon_r \mid r \in R\}, \cdot, 1 \rangle$$

where $\varepsilon_r: A \rightarrow A$ is the unary operation defined by $\varepsilon_r(x) = r \cdot x$ for all $x \in A$. Sometimes we will denote \cdot by juxtaposition. The R -algebra A is a ring under the operations $+, -, 0, \cdot, 1$ and an R -module under the operations $+, -, 0, \{\varepsilon_r \mid r \in R\}$. When considered as a ring, A is precisely the ring R . When considered as an R -module, A is precisely the the ring R considered as an R -module, which we will denote by M or ${}_R R$ when we wish to emphasize that we are considering A (or equivalently R) as an R -module. We will now show that any subset of A that is an ideal when A is considered as a ring forms a submodule when A is considered as an R -module, and conversely. Suppose $X \subseteq A$ is an ideal when A is considered as a ring. Then X is closed under the operations $+, -, 0$. Now let $x \in X$ and $r \in R$. Then we have $\varepsilon_r(x) = rx \in X$, since X is an ideal and hence closed under left multiplication by elements from R . Thus X is closed under the operations $\{\varepsilon_r \mid r \in R\}$. It follows that X is a submodule of M . On the other hand, suppose $X \subseteq A$ is submodule of M . Then X is closed under $+, -, 0$. Let $x \in X$ and $r \in R$. Then we have $rx = \varepsilon_r(x) \in X$ since X is a submodule and hence closed under the operations $\{\varepsilon_r \mid r \in R\}$. It follows that X is closed under left multiplication by elements from R and hence X is an ideal of R .

Consider that the set $\{0\} \subset A$ is the zero ideal when A is considered as a ring and the zero module when A is considered as an R -module. Then if $X \subseteq A$ is an ideal of R with $X \neq (0)$ then X contains a nonzero element and hence $X \neq (0)$ as a submodule of M . Similarly if $X \subseteq A$ is a submodule of M with $X \neq (0)$ then X contains a nonzero element and hence $X \neq (0)$ as an ideal of R . Also, since R, M , and A have the same underlying set R , it follows that a set $X \subset A$ is a proper ideal of R if and only if it is a proper submodule of M . Furthermore if S, T are subsets of A then we have $S \cap T = (0)$ as ideals iff $S \cap T = (0)$ as submodules of M . We also have that the ring $R = S + T$ iff the module $M = S + T$. It follows that S, T are complementary ideals of R iff S, T are complementary submodules of M .

Now suppose R is directly decomposable. That is to say there exist ideals S, T with $(0) \neq S, T \neq R$ such that $R \cong R/S \times R/T$. This is equivalent to the statement that $(0) \neq S, T \neq R$ and S, T are complementary ideals. This holds iff $(0) \neq S, T \neq M$ and S, T are complementary submodules. This is equivalent to the statement that $(0) \neq S, T \neq M$ such that $M \cong M/S \times M/T$. This is precisely the statement that M is directly decomposable. The result follows. \square

- (b) Let M be an R -module. Suppose M is directly decomposable. Then there exist submodules U, V where $(0) \neq U, V \neq M$ such that $M \cong M/U \times M/V$. Then U, V are complementary and hence $M = U + V$ and $(0) = U \cap V$. Then every element $m \in M$ can be written uniquely in the form $m = u + v$ where $u \in U$ and $v \in V$. We claim that the map $\varepsilon: M \rightarrow M$ defined by $\varepsilon(m) = v$ where $m = u + v$, $u \in U$, $v \in V$ is an idempotent endomorphism with $\ker(\varepsilon) \neq 0 \neq \text{im}(\varepsilon)$. To check that this map is an R -module endomorphism, let $m = u + v, m' = u' + v' \in M$ and let $r \in R$. Since U, V are submodules, $u + u' \in U$ and $v + v' \in V$. Then we have that

$$\varepsilon(m + m') = \varepsilon((u + u') + (v + v')) = v + v' = \varepsilon(m) + \varepsilon(m').$$

Since U, V are submodules, $ru \in U$ and $rv \in V$. Thus

$$\varepsilon(rm) = \varepsilon(ru + rv) = rv = r\varepsilon(m).$$

It follows that ε respects addition and scalar multiplication and hence is an R -module endomorphism. Furthermore, we have

$$\varepsilon^2(m) = \varepsilon(\varepsilon(m)) = \varepsilon(v) = \varepsilon(0 + v) = v = \varepsilon(m)$$

and hence ε is idempotent. Suppose $\varepsilon(m) = v = 0$. Then $m = u \in U$. Then $\ker(\varepsilon) \subset U$. On the other hand $\varepsilon(u) = \varepsilon(u + 0) = 0$ so $\ker(\varepsilon) \supset U$. Thus $\ker(\varepsilon) = U \neq 0$. Suppose $m \in \text{im}(\varepsilon)$. Then for some $m' = u' + v'$ we have $m = \varepsilon(m') = v' \in V$. Thus $\text{im}(\varepsilon) \subset V$. On the other hand for all $v \in V$ we have $v = \varepsilon(0 + v)$ so $\text{im}(\varepsilon) \supset V$ and hence $\text{im}(\varepsilon) = V \neq 0$.

Now suppose that there is an idempotent endomorphism $\varepsilon: M \rightarrow M$ such that $\ker(\varepsilon) \neq 0 \neq \text{im}(\varepsilon)$. Suppose $m \in \ker(\varepsilon) \cap \text{im}(\varepsilon)$. Then since $m \in \text{im}(\varepsilon)$ we have that $m = \varepsilon(m')$ for some $m' \in M$. Since $m \in \ker(\varepsilon)$ we have $\varepsilon(m) = 0$. Using the fact that ε is idempotent we compute:

$$m = \varepsilon(m') = \varepsilon^2(m') = \varepsilon(\varepsilon(m')) = \varepsilon(m) = 0.$$

Hence $\ker(\varepsilon) \cap \text{im}(\varepsilon) = (0)$. It follows that $\ker(\varepsilon), \text{im}(\varepsilon) \neq M$, since if either of these submodules were M , the fact that they intersect trivially would mean the other is (0) , contradicting our hypothesis. We now wish to show that $M = \ker(\varepsilon) + \text{im}(\varepsilon)$. We have

$$m = [m - \varepsilon(m)] + [\varepsilon(m)].$$

Set $v = \varepsilon(m)$. Clearly $v \in \text{im}(\varepsilon)$. Then it remains to show that $u = m - \varepsilon(m) \in \ker(\varepsilon)$. We compute

$$\varepsilon(u) = \varepsilon(m - \varepsilon(m)) = \varepsilon(m) - \varepsilon^2(m) = 0$$

since ε is idempotent. Thus we have expressed $m = u + v$ where $u \in \ker(\varepsilon)$ and $v \in \text{im}(\varepsilon)$. It follows that $M = \ker(\varepsilon) + \text{im}(\varepsilon)$. Then we have that $\ker(\varepsilon)$ and $\text{im}(\varepsilon)$ are complementary submodules with $(0) \neq \ker(\varepsilon), \text{im}(\varepsilon) \neq M$. Thus $M \cong M/U \times M/V$ and hence M is directly decomposable. \square

- (c) Let ε be an R -module endomorphism of ${}_R R$. Let $r = \varepsilon(1)$. Let $s \in {}_R R$. Then (in A) we have

$$\varepsilon(s) = \varepsilon(s \cdot 1) = s\varepsilon(1) = sr = rs.$$

Thus $\varepsilon(s) = rs$ for all $s \in {}_R R$. Furthermore, if ε is idempotent, then so is the element r . This is because

$$r^2 = r \cdot r = r \cdot \varepsilon(1) = \varepsilon(\varepsilon(1)) = \varepsilon^2(1) = \varepsilon(1) = r.$$

Thus every R -module endomorphism of ${}_R R$ acts as left multiplication by an element of R and an idempotent endomorphism acts as left multiplication by an idempotent element of R .

- (d) Let $R = R/S \times R/T$ be a direct decomposition of R where S, T are ideals. Then by part (a) we have that $M = M/S \times M/T$ is a direct decomposition of $M = {}_R R$. From part (b) it follows that there is an idempotent R -module endomorphism $\varepsilon: M \rightarrow M$ such that $S = \ker(\varepsilon)$ and $T = \text{im}(\varepsilon)$. From part (c) there exists an idempotent $e \in R$ such that $\varepsilon(x) = ex$ for all $x \in {}_R R$. Then we have

$$T = \text{im}(\varepsilon) = eR = Re = (e)$$

in other words, the image of ε is the ideal generated by e , considered as an R -module. Now suppose $x \in \ker(\varepsilon)$. Then $ex = \varepsilon(x) = 0$. Since $x = ex + (1 - e)x$ we have that $x = (1 - e)x = x(1 - e)$. Then $x \in R(1 - e) = (1 - e)$, the ideal generated by $1 - e$. On the other hand, suppose $x \in (1 - e)$. Then $x = r(1 - e)$ for some $r \in R$. Then we have

$$\varepsilon(x) = \varepsilon(r(1 - e)) = er(1 - e) = r(e - e^2) = 0$$

since e is idempotent. Thus we have

$$S = \ker(\varepsilon) = (1 - e),$$

the ideal generated by $1 - e$, considered as an R -module. Thus $R \cong R/(e) \times R/(1 - e)$ as required.