Math 6150 Commutative Algebra Fall 2009

Assignment 1 Problem 2 **Hill, Hower**

Let R be an integral domain and let K be its field of fractions. Show that the following are equivalent.

(a) For every $x \in K$, either $x \in R$ or $x^{-1} \in R$.

(b) The ideal lattice of R is a chain.

Proof: For future reference, such rings are known as valuation rings. First, assume that for every $x \in K$, either $x \in R$ or $x^{-1} \in R$. Consider any pair of (nonzero) ideals $I, J \lhd R$. Notice that the ideal lattice of R is a chain if and only if $I \subseteq J$ or $J \subseteq I$ for any such pair of ideals. Assume in this case that $I \not\subseteq J$, and we will show $J \subseteq I$. Since $I \not\subseteq J$, there is some $i \in I$ such that $i \notin J$. Let j be any nonzero element of J. The hypothesis states that i/j or j/i is an element of R. If $i/j \in R$, then $i \in (j) \subseteq J$, a contradiction. Thus, we have $j/i \in R$ and hence $j \in (i) \subseteq I$. Since j is arbitrary, we have shown $J \subseteq I$.

Assume now that the ideal lattice of R is a chain. Then, for any nonzero $i, j \in R$ we have $(i) \subseteq (j)$ or $(j) \subseteq (i)$. Consider $i/j \in K$. If $(i) \subseteq (j)$ then $i = rj$ for some $r \in R$, giving $i/j = rj/j = r \in R$. Otherwise, if $(j) \subseteq (i)$ then $0 \neq i/j = i/ir = 1/r$ is an element of K with multiplicative inverse $r \in R$, completing the proof. $\square$