Problem 1. (**Batchelder, Chriestenson, Gern**) *Let k be a field. Describe the ideal lattice of*

- $k[x]$
- $k[[k]]$
- $k((k))$

Solution: We begin with $k[x]$. Recall that since k is a field $k[x]$ is a euclidean domain and hence a principal ideal domain, a unique factorization domain, and an integral domain. Thus every ideal of $k[x]$ is generated some element of $k[x]$.

Define $\equiv$ on $k[x]$ by $p \equiv q \Leftrightarrow p = qu$ where u is unit and $p, q \in k[x]$. Note that for all $p, q, r \in k[x]$ we have that:

- $p \equiv p$ because $p = 1p$
- If $p \equiv q$ then $p = qu$ so $q = pu^{-1}$ hence $q \equiv p$
- If $p \equiv q$ and $q \equiv r$ then there are units u and v in $k[x]$ such that $p = qu$ and $q = rv$; Hence $p = rvu$ and $p \equiv r$ since uv is a unit.

Hence $\equiv$ is an equivalence relation on $k[x]$. Note $p \equiv q$ if and only if the ideals (p) and (q) are equal. Hence the equivalence classes of $\equiv$ are in one to one correspondence with the ideals of $k[x]$. Denote the equivalence class of p by $[p]$.

Define $\leq$ on the equivalence classes of $\equiv$ by $[p] \leq [q] \Leftrightarrow p|q$ in $k[x]$. Note that for any $p, q, r \in k[x]$ we have that:

- $[p] \leq [p]$ since $p = 1p$
- If $[p] \leq [q]$, $[q] \leq [r]$ then $\exists a, b \in k[x]$ such that $q = ap$ and $r = bq$ thus $r = bap$; Ergo $[p] \leq [r]$
- If $p \leq q$ and $q \leq p$ then $\exists a, b \in k[x]$ such that $q = ap$ and $p = bq$ thus $p = bap$ and so $ba = 1$. Thus p and q differ by a unit so $p \equiv q$; Giving $[p] = [q]$.

Therefore we have a partial ordering on the equivalence classes of $\equiv$ in $k[x]$.

For any $p, q \in k[x]$ there are unique integers m and n and unique prime elements $p_i, q_j \in k[x]$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ such that $p = up_1p_2...p_m$ and $q = vq_1q_2...q_n$ where $u, v \in k[x]$ are units. By possibly reordering and renaming the primes found in both decompositions to r_i we can write $p = ur_1r_2...r_tp_{t+1}...p_m$ and $q = vr_1r_2...r_tq_{t+1}...q_n$ for some integer t with $1 \leq t \leq min\{m, n\}$. Now we can define $gcd(p, q) = r_1r_2...r_t$ and $lcm(p, q) = r_1r_2...r_tq_{t+1}...q_mp_{t+1}...p_n$. Clearly $gcd(p, q)|p$ and $gcd(p, q)|q$ and we defined $gcd(p, q)$ to be the largest divisor of both p and q. Likewise $p|lcm(p, q)$ and $q|lcm(p, q)$ and $lcm(p, q)$ is defined to be the smallest element such that p and q both divide it. Thus since prime decomposition is unique up to unit the prime decompositions of $[p]$ and $[q]$ are unique (up to reordering).

Thus we can define $\vee$ and $\wedge$ on the equivalence classes of $\equiv$ by $[p] \vee [q] = [lcm(p, q)]$ and $[p] \wedge [q] = [gcd(p, q)]$. The uniqueness of the prime decompositions shows that these are well defined. Denote by $[k[x]]$ the set of equivalence classes of $\equiv$ in $k[x]$. Then we have that $\langle [k[x]]; \leq, \vee, \wedge \rangle$ is a lattice that is isomorphic to the lattice of ideals of $k[x]$ by construction. We note that if $p, q \in k[x]$ such that $p \leq q$ then the ideal inclusion is reversed; that is $(q) \subseteq (p)$. So the ideal lattice of $k[x]$ looks like A but $\leq$ in A corresponds to $\supseteq$ in the ideal lattice of $k[x]$.

We now move on to $k[[x]]$, the ring of formal power series in the variable x. We recall that the elements of $k[[x]]$ are formal sums $a_0 + a_1x + a_2x^2 + ...$ where $a_i \in k$ for all $i = 0, 1, 2, ...$ Note that it if $A = a_0 + a_1 + a_2x^2 + ...$ and $C = c_0 + c_1x + c_2x^2 + ...$ are inverses of each other then $AB = 1$ hence the 0-th coefficient a_0c_0 must be 1. In particular this means that if A is a unit that a_0 cannot be 0. Now for A given above with $a_0 \neq 0$ we define $B = b_0 + b_1x + b_2x^2 + ...$ by $b_0 = (a_0)^{-1}$ Assuming the first $n-1$ coefficients of B are defined we define b_n in terms of the previous coefficients of B by $b_n = -b_0(b_{n-1}a_1 + b_{n-2}a_2 + ... + b_1a_{n-1} + b_1a_0)$ for each $n \geq 1$. The b_i's are well defined by the principle of recursive definition. Notice that $a_0b_n = b_{n-1}a_1 + b_{n-2}a_2 + ... + b_1a_{n-1} + b_1a_0$ so we get that $a_0b_n + b_{n-1}a_1 + b_{n-2}a_2 + ... + b_1a_{n-1} + b_1a_0 = 0$. This implies $AB = 1$ since for $n \geq 1$ we have $a_0b_n + b_{n-1}a_1 + b_{n-2}a_2 + ... + b_1a_{n-1} + b_1a_0$ is the n-th coefficient of AB; And the 0-th coefficient is 1. Thus A is a unit. Hence every element with nonzero constant term is a unit.

Let I be any ideal on $k[[x]]$ and let l be the smallest integer such that $x^l \in I$. Then $(x^l) \subseteq I \subseteq (x)$. If there is an integer m satisfying $m \leq l$ and $I \cap (x^m) \neq \{0\}$, let $a_0x^l + a_1x^{l+1} + ... \neq 0$ be in $I \cap (x^m)$ such that $a_0 \neq 0$. This implies $x^m(a_0 + a_1x + a_2x^2 + ...) \in I \cap (x^m)$, which is x^m times a unit. Hence x^m is in $I \cap (x^l)$ and thus $x^m = x^l$ so $m = l$. This implies that $I = (x^l)$. Thus every ideal of $k[[x]]$ is generated by a unique element in the set $A = \{x^l | l = 1, 2, ...\}$, likewise every element of this set generates an ideal in $k[[x]]$.

The set A is in bijective correspondence with the strictly positive integers. Note that for positive integers m and n we have that if $m \leq n$ then $(x^n) \subseteq (x^m)$ in $k[[x]]$. Hence A is linearly ordered via this correspondence. Thus the lattice of ideals of $k[[x]]$ is a tower: $\{k[[x]] = (1) \supseteq (x) \supseteq (x^2) \supseteq (x^3) \supseteq ...\}$. More formally it is isomorphic to $\langle A; \leq \rangle$ where $x^n \leq x^m$ in A if and only if $m \leq n$ in $\mathbb{Z}$.

Now we turn to $k((x))$ the ring of formal Laurent series $a_{-n}x^{-x} + ...a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + ...$ for some integer n. First note that that $k[[x]]$ is a subring of $k((x))$ as the sums with no negative indices. Note that since $x^{-1} \in k((x))$ then $xx^{-1} = 1$ and x is a unit is $k((x))$. Hence for any element $f = a_{-n}x^{-x} + ...a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + ... \in k((x))$ we have $(x^n)(f) \in k[[x]]$ and the constant term of $(x^n)(f)$ is nonzero so

$(x^n)(f)$ is a unit is $k[[x]]$. Thus since x is a unit in $k((x))$ we have that f is a unit is $k((x))$.(Note that this argument works with n a negative integer as well.) Since f was chosen arbitrarily we must have that all elements of $k((x))$ are units. Since $k((x))$ is commutative we have that it is a field. Hence the ideal lattice of $k((x))$ lattice is a lattice with two elements: $\{(0) \subseteq k[[x]]\}$. $\square$