**DISCRETE MATH**
**MIDTERM 2**

**Name:**______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. What is the definition of the italicized word or phrase?

(i) $\langle A; \leq \rangle$ is a ***partially ordered set***.

$\langle A; \leq \rangle$ is a *partially ordered set* if $\leq$ is a binary relation on $A$ that is reflexive, antisymmetric and transitive.

(ii) $\langle B; \sqsubseteq \rangle$ is a ***lattice***.

$\langle B; \sqsubseteq \rangle$ is a *lattice* if it is a partially ordered set in which every pair of elements has a greatest lower bound and a least upper bound.

2. Give an example of an infinite partially ordered set that is a lattice, and another that is not a lattice. (Justify your claims about both examples.)

Example 1. The natural numbers under the usual order, $\langle \mathbb{N}; \leq \rangle$, is an infinite lattice. The lattice operations are $\mathrm{glb}(m, n) = \min(m, n)$ and $\mathrm{lub}(m, n) = \max(m, n)$.

Example 2. The natural numbers under the equality relation, $\langle \mathbb{N}; = \rangle$, is an infinite partially ordered set that is not a lattice. The elements 0 and 1 do not have a glb or lub since no element of $\mathbb{N}$ is comparable to both of them in this order.

3. How many partitions of $\{1, 2, \ldots, n\}$ have exactly two cells?

Every partition of $\{1, \ldots, n\}$ into exactly two cells is uniquely determined by specifying the set $U$ of elements which are not in the same cell as the number $n$. $U$ can be any nonempty subset of $\{1, \ldots, n-1\}$, so the number of choices for $U$ is $|\mathcal{P}(n-1)| - 1 = 2^{n-1} - 1$.

4.

(i) Use the Euclidean algorithm to show that $\gcd(21, 100) = 1$.

$$\begin{array}{rll} 100 & = q_1 * 21 + r_1 & = 4 * 21 + 16 \\ 21 & = q_2 * 16 + r_2 & = 1 * 16 + 5 \\ 16 & = q_3 * 5 + r_3 & = 3 * 5 + 1 \\ 5 & = q_4 * 1 + r_4 & = 5 * 1 + 0, \end{array}$$

so the remainder sequence is $(100, 21, 16, 5, 1, 0)$. The last nonzero term is $1 = \gcd(100, 21)$.

(ii) Find $a$ and $b$ such that $21a + 100b = 1$.

Using equations from (i) we get

$$\begin{array}{rll} 1 & = 16 - 3 * (5) & \\ & = 16 - 3 * (21 - 1 * 16) & = -3 * 21 + 4 * (16) \\ & & = -3 * 21 + 4 * (100 - 4 * 21) = -19 * 21 + 4 * 100, \end{array}$$

so $a = -19$ and $b = 4$.

(iii) Find the inverse of $\overline{21}$ in $\mathbb{Z}_{100}$ and the inverse of $\overline{100}$ in $\mathbb{Z}_{21}$. (The answer to the first question should be in the form $\overline{k}$ for some $k \in \{0, 1, \ldots, 99\}$, and the answer to the second question should be in the form $\overline{k}$ for some $k \in \{0, 1, \ldots, 20\}$.)

Since $-19 * 21 + 4 * 100 = 1$, we get $\overline{1} = -\overline{19} * \overline{21} + \overline{4} * \overline{100} = \overline{4} * \overline{100}$ in $\mathbb{Z}_{21}$. Thus $\overline{100}^{-1} = \overline{4}$ in $\mathbb{Z}_{21}$.

Since $\overline{1} = -\overline{19} * \overline{21} + \overline{4} * \overline{100} = -\overline{19} * \overline{21} = \overline{81} * \overline{21}$ in $\mathbb{Z}_{100}$, we get $\overline{21}^{-1} = \overline{81}$ in $\mathbb{Z}_{100}$.