DISCRETE MATH
MIDTERM 1

Name:____________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. What is the definition of the italicized word or phrase?

   (a) "f is a ***function*** from A to B".

   f is a ***function*** from A to B if $f \subseteq A \times B$ and for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

   (b) "$\mathcal{P}(X)$ is the ***power set*** of X".

   The ***power set*** of X is the set $\mathcal{P}(X)$ of all subsets of X.

2. Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

$$
\begin{aligned}
x \in \mathcal{P}(A \cap B) \quad &\Leftrightarrow x \subseteq A \cap B \\
&\Leftrightarrow x \subseteq A \ \text{ and } \ x \subseteq B \\
&\Leftrightarrow x \in \mathcal{P}(A) \text{ and } x \in \mathcal{P}(B) \\
&\Leftrightarrow x \in \mathcal{P}(A) \cap \mathcal{P}(B)
\end{aligned}
$$

3.

(a) What is the recursive definition of multiplication of natural numbers?

$$\begin{array}{lrl} \text{(IV)} & m \cdot 0 & := 0 \\ \text{(RR)} & m \cdot S(n) & := m \cdot n + m \end{array}$$

(b) Prove by induction that $\ell(m+n) = \ell m + \ell n$ for all $\ell, m, n \in \mathbb{N}$. (You may use any previously proved theorems that concern *addition*.)

Let S_n be the statement that "$\ell \cdot (m+n) = \ell \cdot m + \ell \cdot n$ for all $\ell, m \in \mathbb{N}$".

(Basis of induction) We must show that S_0 is true, that is that $\ell \cdot (m+0) = \ell \cdot m + \ell \cdot 0$ holds for all $\ell, m \in \mathbb{N}$.

$$\begin{array}{rll} \ell \cdot (m+0) & = \ell \cdot m & ((\text{IV}), +) \\ & = \ell \cdot m + 0 & ((\text{IV}), +) \\ & = \ell \cdot m + \ell \cdot 0 & ((\text{IV}), \cdot). \end{array}$$

(Inductive step) Assume S_n is true. We must show that S_{n+1} is true, which means that $\ell \cdot (m + S(n)) = \ell \cdot m + \ell \cdot S(n)$ for all $\ell, m \in \mathbb{N}$.

$$\begin{array}{rll} \ell \cdot (m+S(n)) & = \ell \cdot S(m+n) & ((\text{RR}), +) \\ & = \ell \cdot (m+n) + \ell & ((\text{RR}), \cdot) \\ & = (\ell \cdot m + \ell \cdot n) + \ell & (\text{Inductive Hypothesis}) \\ & = \ell \cdot m + (\ell \cdot n + \ell) & (\text{Associative Law}, +) \\ & = \ell \cdot m + \ell \cdot S(n) & ((\text{RR}), \cdot). \end{array}$$

4. The set $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ has $2n$ elements.

(a) Justify the following claim: the number of n-element subsets of X that consist of k of the a's and $n-k$ of the b's is $\binom{n}{k}\binom{n}{n-k}$.

According to the definition of the binomial coefficients, the number of ways to choose k of the a's is $\binom{n}{k}$ and the number of ways to choose $n-k$ of the b's is $\binom{n}{n-k}$. By the Product Rule, the number of ways to do both is given by the product $\binom{n}{k}\binom{n}{n-k}$.

(b) Explain why the number of n-element subsets of X is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}.$$

Every n-element subset of X contains some number k of the a's and then $n-k$ of the b's. By part (a) and the Sum Rule, the number of ways of selecting an n-element subset is therefore $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.

(c) Explain why

$$\begin{aligned}\binom{2n}{n} &= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}\\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.\end{aligned}$$

The first equality holds because the left hand side is defined to be the number of n-element subsets of a $2n$-element set, and the right hand side was shown in (b) to be the same number. The second equality holds because $\binom{n}{k} = \binom{n}{n-k}$ for every k. Hence $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$ for every k.