Discrete Math (MATH 2001)
HANDOUT 2 (October 8, 2008)

PROBLEM SOLVING

Faced with a strange new math problem? Not sure how to start? You're in good company! There are problems around that no mathematician has been able to solve for hundreds of years. This does not mean problem solving is hopeless, because not all problems are hard. You can improve your chance of successfully solving a math problem by doing some obvious things.

Suggestion 1. Make sure you understand what the problem is asking.

For example, if you read:

3. Let e_n be the number of equivalence relations on an n-element set.
   (a) Find e_0, e_1, e_2, e_3, e_4.
   (b) Show that e_n is between 2^n and 2^{n^2} if $n > 4$.

then what is e_n? Is it an equivalence relation? Is it a partition? Know all terms, and understand how they fit together.

Suggestion 2. Make sure you know what constitutes a solution.

In the problem mentioned under Suggestion 1, what are the legitimate ways to show that one number lies between two others?

Suggestion 3. Draw a picture if possible, or work out small cases.

In the problem mentioned under Suggestion 1, you might try to write down all equivalence relations on a set of size $n = 3$ or 4, and see if it gives you an idea for a solution. For comparison, you might also want to write down $\mathcal{P}(n)$ and $\mathcal{P}(n \times n)$ for $n = 3$ or 4.

Suggestion 4. Try to look at the problem in different ways.

In the problem mentioned under Suggestion 1, you might consider the number of partitions of an n-element set instead of the number of equivalence relations. It turns our that it is easier to prove $2^n \leq e_n$ if you think of e_n is the number of partitions of an n-element set, while it is easier to prove $e_n \leq 2^{n^2}$ if you think of e_n is the number of equivalence relations on an n-element set.

Suggestion 5. Make use of experience.

Have you seen similar problems before? Do you know any relevant theorems? Is there a book, person or website that might provide relevant information?

Today we will practice solving problems related to the number e_n. For this exercise, I'll use more standard terminology: the number of equivalence relations on n is B_n, the nth *Bell number*. The Bell number are named after Eric Temple Bell, who is best known today for his book of biographical sketches, called *Men of Mathematics*. (So $B_n = e_n$.)

Problems.

(1) Determine how the numbers $2^{n-1}, B_n, n!, 2^{n^2}$ are related to each other as n grows. (Which is larger than which?)

(2) Show that the Bell numbers are equal to the sequence defined recursively by

$$\begin{aligned} B_0 &= 1 \\ B_{n+1} &= \binom{n}{n} B_n + \binom{n}{n-1} B_{n-1} + \cdots + \binom{n}{0} B_0. \end{aligned}$$

(3) Show that there are $n!$ ways to linearly order the numbers $1, 2, \ldots, n$.