ALGEBRA TEST #3

This exam is due Monday, December 18, at 10am. Do four of the problems. You may use your book, but you may not communicate with others concerning the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

I have neither given nor received aid on this exam.

Name:______________________________

1. (Alternative definition of k-algebra.) Let k be a commutative ring.
   (a) Let $\mathbf{A} = \langle A; \cdot, 1, +, -, 0, \{r(x) \mid r \in k\}\rangle$ be a k-algebra. Show that the function $\varphi\colon k \to \mathbf{A}\colon r \mapsto r(1)$ is a ring homomorphism from k into the center of $\mathbf{A}$. (Which is $Z(\mathbf{A}) = \{a \in A \mid \forall x \in A(ax = xa)\}$.)
   (b) Show conversely that any ring homomorphism $\varphi\colon k \to \mathbf{A}$ from k into the center of some ring $\mathbf{A}$ defines a unique k-algebra structure on $\mathbf{A}$, via $r(a) = \varphi(r) \cdot a$ whenever $r \in k$ and $a \in A$.
   (c) Show that if ring homomorphisms $\varphi\colon k \to \mathbf{A}$ and $\psi\colon k \to \mathbf{B}$ define k-algebra structures on $\mathbf{A}$ and $\mathbf{B}$, then a ring homomorphism $\lambda\colon \mathbf{A} \to \mathbf{B}$ is a k-algebra homomorphism iff $\lambda \circ \varphi = \psi$.

2. Prove that a surjective endomorphism of a Noetherian ring is an isomorphism.

3. Show that the intersection of any chain of prime ideals in a commutative ring is prime. Conclude that if $I \lhd R$ is a proper ideal, then there exists a minimal element in the set of prime ideals containing I.

4. Do Exercise 8.3.8 of Dummit and Foote.

5. Let $\omega = e^{2\pi i/3}$ be a primitive third root of unity. Show that $\mathbb{Z}[\omega]$ is a Euclidean Domain. Show that $1 - \omega$ is a prime element.

6. Use the fact that $\mathbb{Z}[i]$ is a unique factorization domain to find all integer solutions to $x^2 + y^2 = z^3$ that satisfy $\gcd(x, y) = 1$.

7. Let $\mathbb{F}$ be a field. Show that $\mathbb{F}[x]$ has an infinite set of pairwise nonassociate primes.

8. Decide (with proof) whether r is irreducible in the integral domain $\mathbf{D}$:
   (a) $r = x^n + px + p^2$, $\mathbf{D} = \mathbb{Z}[x]$, p is a prime integer.
   (b) $r = x^2 + y^2 - 1$, $\mathbf{D} = \mathbb{F}[x, y]$, $\mathbb{F}$ is a field.
   (c) $r = x^4 - 10x^2 + 1$, $\mathbf{D} = \mathbb{Z}_p[x]$, p an unspecified prime. (I.e., decide the answer for each p.)