ALGEBRA TEST #2

This exam is due Friday, October 31. Do three of the problems. You may use your book, but you may not communicate with others concerning the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

I have neither given nor received aid on this exam.

Name:____________________________

1. Show that if N is a minimal normal subgroup of some finite group G, then N is isomorphic to a power S^k of a simple group S.

2. Describe the structural properties a group G must have if the free G-set $\langle G; G\rangle$ has a nontrivial direct factorization (meaning $\langle G; G\rangle \cong \mathbf{A} \times \mathbf{B}$ with $|A|, |B| > 1$). Then find a nonabelian finite group G such that $\langle G; G\rangle$ has no nontrivial direct factorization.

3. Let $\mathbf{F}_{Grp}(a, b)$ be the group freely generated by the set $X = \{a, b\}$. Show that the subset $\{ab, a^2b^2, a^3b^3, \ldots\}$ is an infinite independent subset of this group. Conclude that a free group generated by a countably infinite set is embeddable in a free group on two generators.

4. In this problem you will establish a fairly compact presentation of the symmetric group S_{n+1}. Take as generators $\tau_1, \ldots, \tau_n$ and as relations

- $\tau_i^2 = 1$ for all i.
- $(\tau_i\tau_{i+1})^3 = 1$ for all i (equivalently, $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ for all i).
- $\tau_i\tau_j = \tau_j\tau_i$ if $|j - i| > 1$.

(a) Show that the group with this presentation has a homomorphism onto S_{n+1}.

(b) Show that the group with this presentation has at most $(n + 1)!$ elements.

(Hint for (b): Use the relations and induction on n to show that any product of generators may be rewritten as a product of the same length or shorter in the form $\sigma\tau_n\tau_{n-1}\cdots\tau_{k+1}\tau_k$ for some k where σ is a product of generators not including τ_n.)

5. Let $\mathbf{A} = \langle G/H; G\rangle$ be the G-set of left cosets of H under the action of left multiplication. Show that $\mathrm{Aut}(\mathbf{A}) \cong N_G(H)/H$.

6. Show that a coproduct of a family $\{\mathbf{A}_j \mid j \in J\}$ of G-sets is $(\mathbf{A}, \{\iota_j\})$ where $\mathbf{A}$ is the disjoint union of the $\mathbf{A}_j$ and ι_j is inclusion.