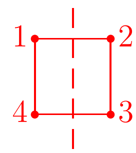
ALGEBRA 1
MIDTERM 1

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Draw a regular 4-gon and label the vertices with the numbers $1, 2, 3, 4$.



(a) Write the cycle decomposition of each element of $D_4 = \{1, r, r^2, r^3, f, rf, r^2f, r^3f\}$.

$$\begin{aligned} 1 &= 1 & f &= (1\ 2)(3\ 4) \\ r &= (1\ 2\ 3\ 4) & rf &= (1\ 3) \\ r^2 &= (1\ 3)(2\ 4) & r^2f &= (1\ 4)(2\ 3) \\ r^3 &= (1\ 4\ 3\ 2) & r^3f &= (2\ 4) \end{aligned}$$

(b) Is it possible to label the edges with $1, 2, 3, 4$ so that the cycle decomposition of each permutation in D_4 is the same as it was in (a)? (Show or explain.)

No. Depending on the axis you choose for the flip f, either f fixes a vertex and no edge or it fixes an edge but no vertex. Either way, f will have 1-cycles under one of its actions but not under the other action.

(c) For which m (if any) do there exist labelings of the vertices and the edges of a regular m-gon with $1, \ldots, m$ so that the cycle decomposition of any permutation relative to the vertex labeling is the same as the cycle decomposition relative to the edge labeling?

If m is even, then the axis of the flip f either passes through two opposite vertices or pierces the middle of two opposite sides. If it passes through opposite vertices, then it fixes two vertices and no sides. If the axis pierces opposite sides, then it fixes two sides and no vertices. In either case, the number of fixed points (= 1-cycles) of f differs, so the cycle types differ.

But if m is odd, then the vertices and edges can be labeled so that the cycle types of all rigid motions are the same with repect to the two labelings: just label so that each edge label matches the label on the opposite vertex.

2. If G is a group and $m, n \in G$, then the *commutator of* m *and* n is $[m, n] := m^{-1}n^{-1}mn$. If M and N are subgroups, the *commutator of* M *and* N is subgroup generated by $\{[m, n] \mid m \in M, n \in N\}$.

(a) Show that $[M, N] = \{1\}$ iff every element of M commutes with every element of N.

$[M, N] = \{1\}$ iff each of its generators equals 1 iff $m^{-1}n^{-1}mn = [m, n] = 1$ when $m \in M$ and $n \in N$ iff $mn = nm$ when $m \in M$ and $n \in N$.

(b) Show that a subgroup $N \leq G$ is normal iff $[G, N] \subseteq N$.

$N \lhd G$ iff for all $g \in G$ and $n^{-1} \in N$ it is the case that $g^{-1}n^{-1}g \in N$. But since $n, n^{-1} \in N$, we have $g^{-1}n^{-1}g \in N$ iff $g^{-1}n^{-1}gn \in N$. Thus, $N \lhd G$ iff $[g, n] \in N$ for all $n \in N$ and $g \in G$. The result now follows from the fact that $[G, N] \subseteq N$ iff the generators of $[G, N]$ lie in N.

(c) Show that if N is a normal subgroup of S_n, then either $[S_n, N] = \{1\}$ or N contains an element that is a product of exactly two transpositions. (Use the fact that S_n is generated by transpositions.)

The problem should have said "exactly two distinct transpositions", otherwise the problem is trivial.

If $[S_n, N] \neq \{1\}$, then some $\tau \in S_n$ fails to commute with some $\nu \in N$. We may assume that τ is a transposition, since the transpositions generate S_n, and if ν commutes with a set of generators of S_n it commutes with every element of S_n. Hence $1 \neq [\tau, \nu] \in [S_n, N] \subseteq N$, implying that N contains $\tau^{-1}\nu^{-1}\tau\nu \neq 1$. But $\tau^{-1} \cdot (\nu^{-1}\tau\nu)$ is a product of two transpositions, since inverses and conjugates of transpositions are transpositions. The transpositions are distinct since their product is not 1.

(d) Show that if N is a normal subgroup of S_n, $n > 4$, and N contains an element that is a product of exactly two transpositions, then N contains a 3-cycle.

N contains a product of two distinct transpositions, so contains an element whose cycle decomposition is $(a\ b)(c\ d)$ or else is $(a\ b)(b\ c) = (a\ b\ c)$. In the latter case there is nothing to prove, while in the former case we can choose $e \notin \{a, b, c, d\}$ (since $n > 4$) and conjugate $\sigma = (a\ b)(c\ d) \in N$ by $\tau = (d\ e)$. We obtain $\tau\sigma\tau^{-1} = (d\ e)(a\ b)(c\ d)(d\ e) = (a\ b)(c\ e) \in N$. Now the ratio $(\tau\sigma\tau^{-1})\sigma^{-1} = (c\ d\ e)$ is a 3-cycle in N.