What is a Universal Property?

Modern Algebra 1

Fall 2008

The Algebraization of Functional Composition

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(3) *Any class of algebras equipped with all algebra homomorphisms is a category.*

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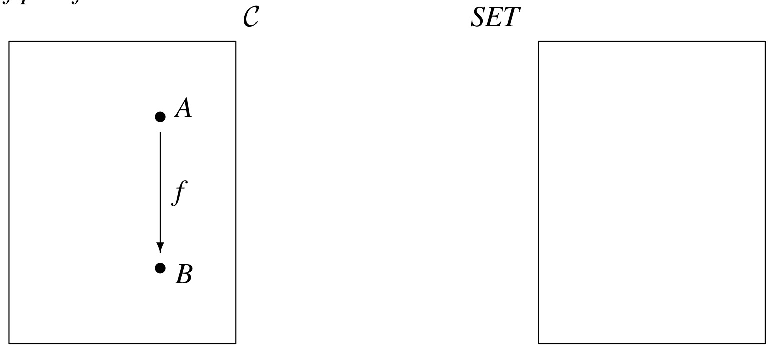

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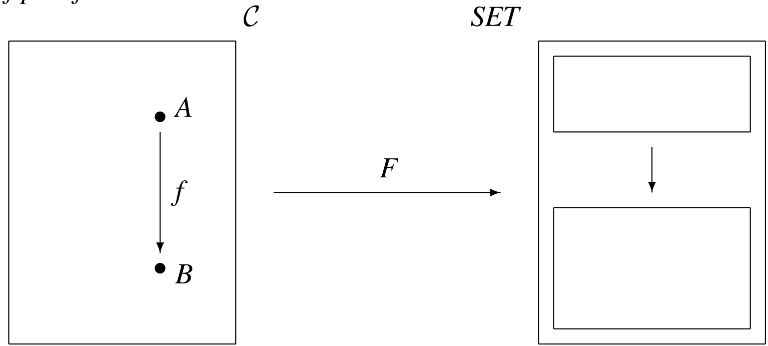

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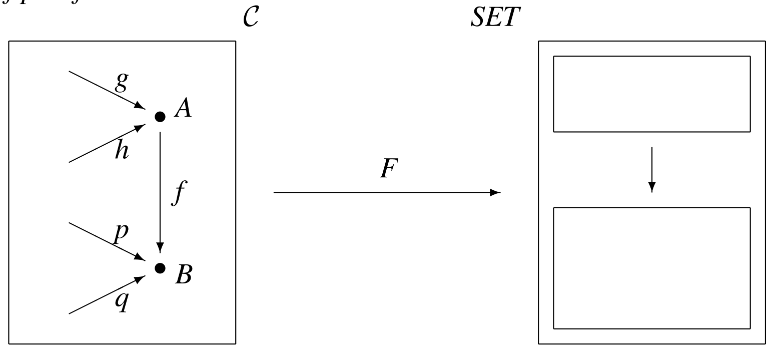

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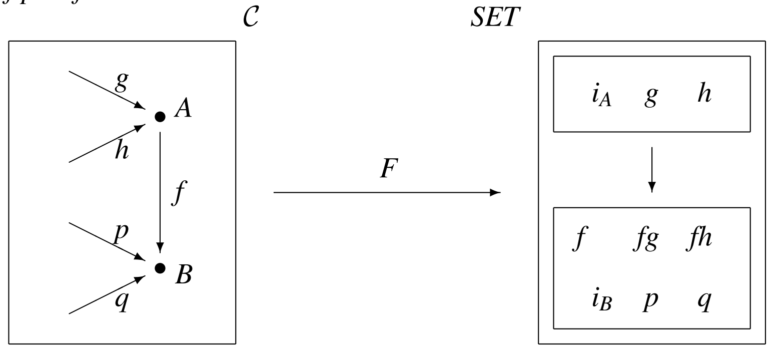

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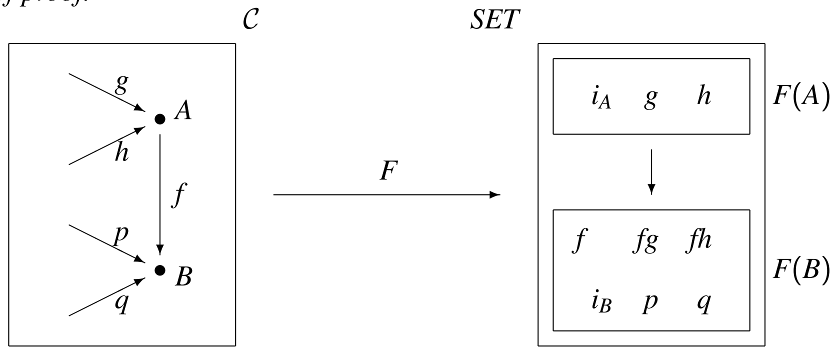

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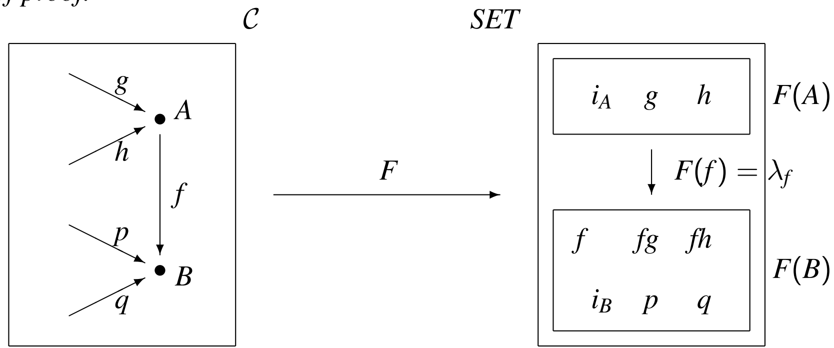

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Universal Morphisms

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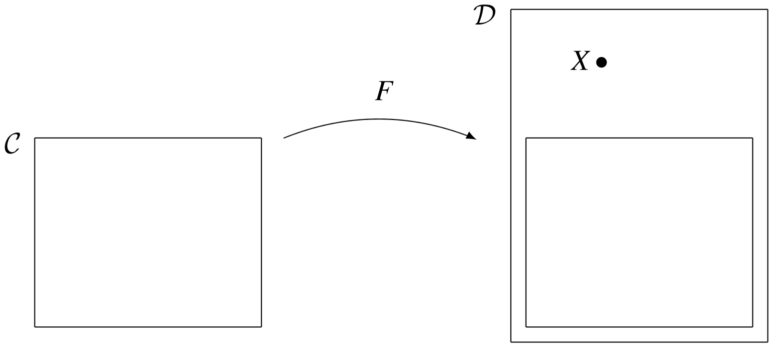

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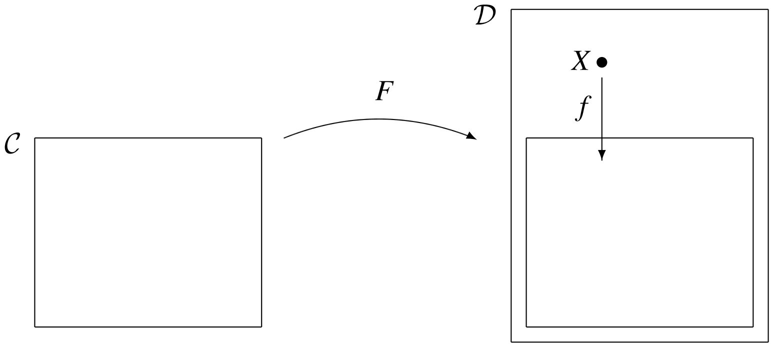

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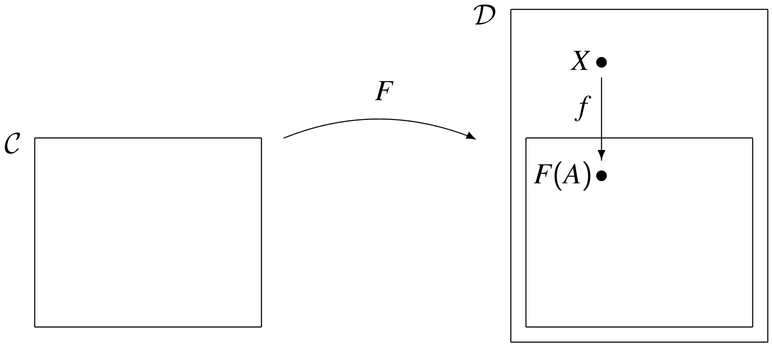

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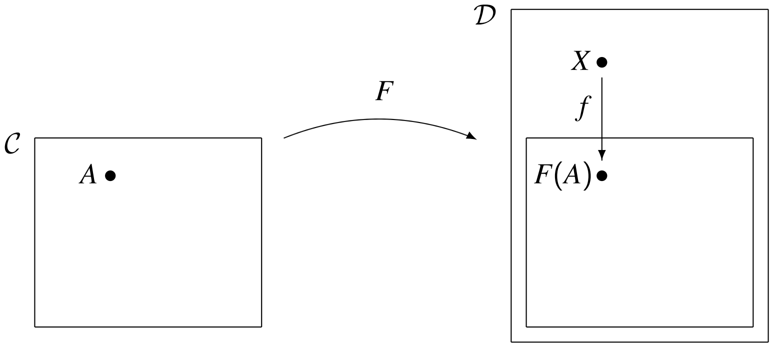

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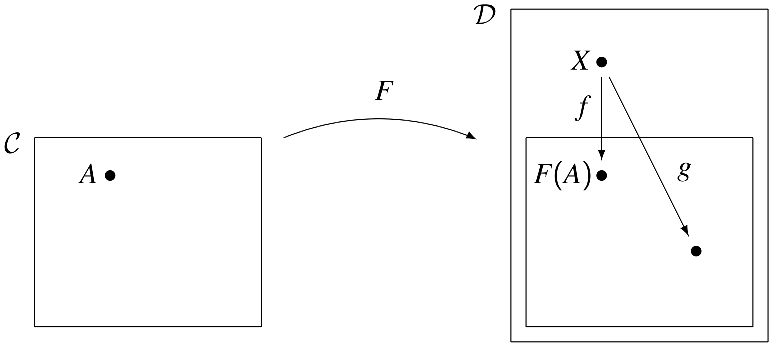

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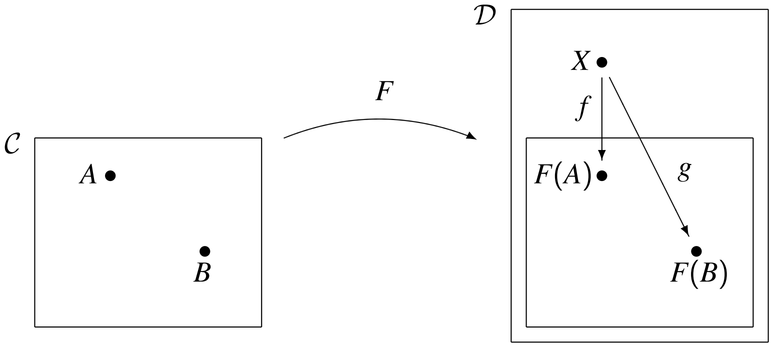

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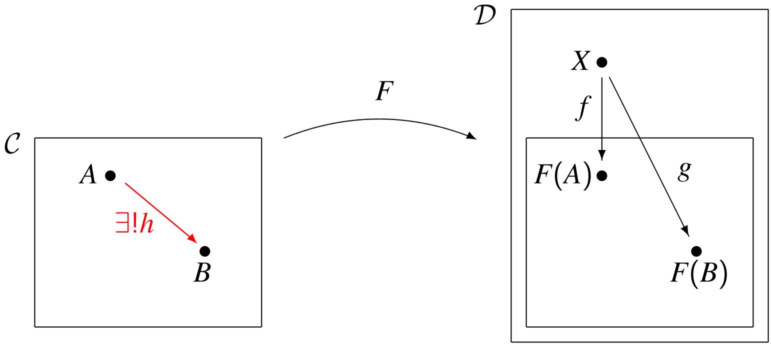

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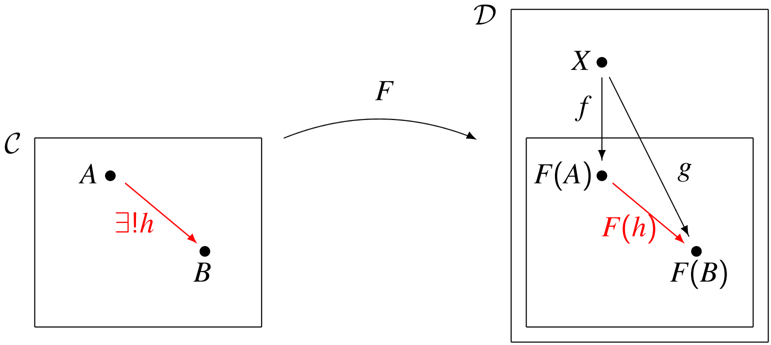

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