Modern Algebra 1 (MATH 6130)
HANDOUT 4 (September 24, 2008)

SUMMARY OF TOPICS FROM 8/25/08-9/24/08

I. Introduction (Week 1)
  (i) What is "Algebra" about?
  (ii) What is an algebra?
  (iii) Where do algebras come from? What are they for? How can they be used to solve problems?

II. The Algebraization Process (Week 1)
  (i) Identify a situation where an algebraic model is appropriate.
  (ii) Choose a language.
  (iii) Side issue: equivalence of languages.
  (iv) Determine axioms.
  (v) Test axioms for correctness and completeness.

III. Example: The Cayley Representation (Week 1)
  (i) Cayley representation: $a \mapsto \lambda_a$, $\lambda_a(x) := ax$. Also called the (left) regular representation.
  (ii) Compare with the right regular representation: $a \mapsto \rho_a$, $\rho_a(x) := xa$. This should be used when functions act on the right.
  (iii) Variants of this proof establish representation theorems for semigroups, monoids, groups, rings, k-algebras, clones and categories.

IV. Compatible Relations (Week 2)
  (i) If $\mathbf{A}_i$ have the same language, then a relation $R \subseteq \prod A_i$ is compatible if it is closed under componentwise application of operations. Once later definitions are in place, this just says that R is a subuniverse of the product $\prod \mathbf{A}_i$.
  (ii) Examples: compatible function =: homomorphism, compatible unary relation =: subuniverse, compatible equivalence relation =: congruence.
  (iii) Significance: the compatible relations of an algebra are invariant with respect to a change to any equivalent language. (The subgroups, congruences, etc. of a given multiplication group are the same when it is considered as a division group.) In fact, the class of compatible relations determines an algebra up to choice of language. (Any algebra that has the same compatible relations as a group G must be G itself in some equivalent language.)
  (iv) The collection of compatible relations on a given algebra $\mathbf{A}$ contains the binary relation equality $= \{(x,x) \in A^2 \mid x \in A\}$, and is closed under the following constructions:
    (a) projection onto a subset of coordinates,
    (b) permutation of coordinates,

(c) product (R and S compatible implies $R \times S$ compatible),
(d) intersection of relations of the same arity,
(e) up-directed union.

V. A Fundamental Picture (Week 2)

A picture goes here.
(You know which one.)

(i) There is a unique algebra structure on $\mathrm{im}(\varphi)$ that makes the inclusion map $\mathrm{id}_{\mathrm{im}(\varphi)}$ a homomorphism. The image of $\mathrm{id}_{\mathrm{im}(\varphi)}$ is the same as the image of φ.
(ii) There is a unique algebra structure on $A/\ker(\varphi)$ that makes the natural map ν a homomorphism. The kernel of ν is the same as the kernel of φ.
(iii) Side comment: a bijective homomorphism is an isomorphism. This is a theorem of algebra whose analog fails for ordered sets or topological spaces.

VI. Groups. The Symmetric Group (Week 2)
(i) Examples of groups. (Cyclic groups, S_n, D_n, $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$.)
(ii) Representations of permutations. Support of a permutation. Fixed points.
(iii) Transpositions. Cycle type. Cauchy number. Sign of a permutation. S_n is generated by permutations.
(iv) Definition of the alternating group.
(v) Which sets of transpositions generate S_n?
(vi) Normal subgroups abbreviate group congruences. The correspondence between them is part of the correspondence between subgroups and right invariant equivalence relations.

VII. Subalgebra Generation (Week 3)
(i) Subuniverse generation can be described from bottom up, as an iterative process, or top down, as an intersection.
(ii) Subalgebra generation is (the prototypical example of) an algebraic closure operator.

VIII. Lattices (Week 3)
(i) Axioms. (Lattices are algebraic model of ordered sets under lub and glb.)
(ii) Complete lattices, algebraic lattices.
(iii) $\mathrm{Sub}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ are algebraic lattices.
(iv) Some algebraic lattices do not arise as $\mathrm{Sub}(G)$, G a group.
(v) Modular lattices. $\mathrm{Norm}(G)$ is modular.

IX. Index (Week 4)
(i) Index can be attached to subgroup lattices to convey more information.
(ii) Index of subgroup is the number of right cosets (= number of left cosets).
(iii) $|K| = [K : 1]$, $[G : K][K : H] = [G : H]$ if $H \leq K \leq G$, $|HK| \cdot |H \cap K| = |H| \cdot |K|$ if H and K are finite, $[H \vee K : K] \geq [H : H \cap K]$ with equality iff $HK = KH$.

(iv) Lagrange's Theorem, classification of groups of very small order.

X. Isomorphism Theorems (Week 4)

(i) The First Isomorphism Theorem is what is depicted in the Fundamental Picture. Analog fails for ordered sets or topological spaces. ("Fails" really means a significantly less useful conclusion holds.) a bijective homomorphism is an isomorphism.

(ii) The Second Isomorphism Theorem: If $\varphi\colon \mathbf{A} \to \mathbf{B}$ is a hom, $\theta = \ker(\varphi)$, and $\mathbf{S} \leq \mathbf{A}$, then $\mathbf{S}/\theta|_{\mathbf{S}} \cong \mathbf{S}^\theta/\theta|_{\mathbf{S}^\theta}$.