Modern Algebra 1 (MATH 6130)
HANDOUT 1 (September 3, 2008)

TERMINOLOGY

Terminology about functions.

Definition 1. Let $f\colon A \to B$ be a function from a set A to a set B.

(1) The *image* of f is the set $\mathrm{im}(f) = \pi_2(f) = \{b \in B \mid \exists a(b = f(a))\}$.
(2) The *kernel* of f is the binary relation $\ker(f) = \{(x, y) \in A^2 \mid f(x) = f(y)\}$.
(3) If θ be an equivalence relation on A, then the *quotient set*, A/θ, is the set $\{a/\theta \mid a \in A\}$ of all cosets of θ. When $\theta = \ker(f)$ for some f, then $A/\ker(f)$ is called the *coimage* of f.
(4) $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is called the *natural map* associated to θ.
(5) If $C \subseteq B$, then the identity map, $c \mapsto c$, on C, considered as a function to B, is called the *inclusion map*.

Facts 2. Let $f\colon A \to B$ be a function from a set A to a set B.

(1) The *kernel* of f is an equivalence relation. Conversely, any equivalence relation θ on A is the kernel of some function (e.g., the natural map associated to θ).
(2) The image of f is a subset of B. Conversely every subset $C \subseteq B$ is the image of some function (e.g., the inclusion map associated to C).
(3) The natural map associated to any equivalence relation is surjective. The inclusion map associated to any subset is injective.
(4) Given $f\colon A \to B$, the relation $\overline{f} = \mathrm{id}_{\mathrm{im}(f)}^{-1} \circ f \circ \nu^{-1}$ is a bijection from $A/\ker(f) \to \mathrm{im}(f)$ whose inverse is the relation $\nu \circ f^{-1} \circ \mathrm{id}_{\mathrm{im}(f)}$. $\bar{f}$ is called the *induced map*.

Terminology about equivalence relations.

Definition 3. (1) If θ is an equivalence relation on A, then a subset $S \subseteq A$ is *θ-saturated* if it is a union of θ-classes.
(2) If θ is an equivalence relation on A, then the *θ-saturation* of a subset $S \subseteq A$, written S^θ is the smallest θ-saturated subset of A containing S.

Facts 4. Let $f\colon A \to B$ be a function from a set A to a set B.

(1) If $\theta = \ker(f)$, then a subset $S \subseteq A$ is θ-saturated iff $S = f^{-1}(T)$ for some subset $T \subseteq B$.
(2) The θ-saturation of $S \subseteq A$ is $S^\theta = \{x \in A \mid \exists s \in S(x \equiv s \pmod{\theta})\}$.