ALGEBRA 1
HOMEWORK ASSIGNMENT VII

(Turn in underlined problems.)

PROBLEMS

1. Show that an arbitrary coproduct of free algebras is free.

2. You have shown that the coproduct of two abelian groups is isomorphic to their cartesian product equipped with certain coprojection maps. This does not hold for infinite coproducts (see Exercise 10.3.24). Show instead that if $\{A_j \mid j \in J\}$ is a family of abelian groups, then a coproduct of this family is $(A, \{i_j\})$ where A is the subgroup of $\prod A_j$ consisting of all tuples that are zero almost everywhere and $i_j \colon A_j \to A$ is the homomorphism that maps $a \in A_j$ to the tuple $(\ldots, 0, 0, a, 0, 0, \ldots)$, with a in the j-th position.

3. An algebra is *finitely generated* if it has a finite generating set, and *finitely presentable* if it has a presentation $\langle X \mid R \rangle$ with X and R both finite.

(a) Show that if A is finitely generated and Y is any generating set for A, then there is a finite subset $Y_0 \subseteq Y$ that is a generating set for A.

(b) Show that if A is finitely presentable and $\langle Y \mid S \rangle$ is any presentation of A with Y finite, then there is a finite subset $S_0 \subseteq S$ such that $\langle Y \mid S_0 \rangle$ is a presentation of A.

<u>4.</u> Show that if $\varphi \colon \mathrm{A} \to \mathrm{B}$ is a surjective homomorphism, where A is finitely generated and B is finitely presentable, then $\ker(\varphi)$ is finitely generated.

<u>5.</u> Show that a finite algebra in a finite language is finitely presentable. (A language is finite if it has finitely many basic operation symbols.)

<u>6.</u> Suppose that G is a finitely generated infinite group. Show that G has a normal subgroup N such that G/N is infinite, but all normal subgroups of G/N have finite index. Give an example to show that this statement may be false if G is not finitely generated.

Challenge Problem #7 (Prize = 1 coin) Let A_n be the abelian group presented by $\langle X \mid R \rangle$ where $X = \{x_0, \ldots, x_{n-1}\}$ and R consists of all relations of the form $x_i \cdot x_{i+1} = x_{i+2}$ where the subscripts are taken modulo n. Show that A_n is isomorphic to a group of the form (1) $\mathbb{Z}_m$, (2) $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$, (3) $\mathbb{Z}_m \times \mathbb{Z}_m$, or (4) $\mathbb{Z}_m \times \mathbb{Z}_{5m}$.