ALGEBRA 1
HOMEWORK ASSIGNMENT VI

(Turn in underlined problems.)

PROBLEMS

1. Let A be an abelian group of exponent p^k for some prime p and some positive integer k.

(a) Show that there is an element $u \in A$ of order p^k.

(b) Given u as in (a), choose a subgroup $N \leq A$ that is maximal for the property that $N \cap \langle u \rangle = \{0\}$. Show that $A \cong N \times \langle u \rangle$.

(c) Show that any finite abelian group is a direct product of cyclic groups.

(Hint for (b): Show that A/N has a unique minimal subgroup, and that for abelian p-groups this implies that A/N is cyclic. Show that A/N must be cyclic of order p^k, and that the elements of $\langle u \rangle$ must represent the different cosets of N. Conclude from this that $A = N\langle u \rangle$, and then that $A \cong N \times \langle u \rangle$.)

2. (Coproducts of abelian groups) Let A and B be abelian groups, written additively. Define *coprojections* $i_A \colon A \to A \times B \colon a \mapsto (a, 0)$ and $i_B \colon B \to A \times B \colon b \mapsto (0, b)$. Show that $(A \times B, i_A, i_B)$ is a coproduct of A and B relative to the category of all abelian groups.

3. Show that the coproduct of $\mathbb{Z}_2$ with $\mathbb{Z}_2$ relative to the category of abelian groups is not the same as the coproduct relative to the category of all groups.

4. The *abelianization* of a group G is the group $G/[G, G]$.

(a) Show that the abelianization of G is abelian.

(b) Let Grp be the category of all groups equipped with all homomorphisms, and let Ab be the subcategory of abelian groups equipped with all homomorphisms. The inclusion $I \colon Ab \to Grp$ is a functor (you may assume). Show that if G is any group, then a universal morphism from G to I is $(G/[G, G], \nu)$, where $\nu \colon G \to G/[G, G]$ is the natural map.

5. Show that there is no functor from Grp to Ab whose object part maps every group to its center. (Hint: Suppose that F was such a functor. Explore what F must do to any embedding $\varphi\colon \mathbb{Z}_2 \to S_3$.)

6. Suppose that P and Q are ordered sets, considered as categories, and that $F\colon P \to Q$ is a functor. Explain the meaning of the following concepts in order-theoretic terms.

(a) Every pair of elements of P has a product (coproduct).
(b) (A, f) is a universal morphism from X to F (to X from F).

7. Give an example of a category $\mathcal{C}$ and a subcategory $\mathcal{D}$ such that, in either category, any pair of objects has a product and a coproduct, *but* some pair of objects A, B of $\mathcal{D}$ has $A \sqcap_{\mathcal{C}} B \ncong A \sqcap_{\mathcal{D}} B$ and $A \sqcup_{\mathcal{C}} B \ncong A \sqcup_{\mathcal{D}} B$.

Challenge Problem #6 (Prize = 1 coin)

Let $G(n)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices $U(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ and $L(n) = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$. Show that $G(2)$ is free over $\{U(2), L(2)\}$, but $G(1)$ is not free over $\{U(1), L(1)\}$.