ALGEBRA 1
HOMEWORK ASSIGNMENT IV

(Turn in underlined problems.)

Read Sections 3.1–3.3.

SECTION	PROBLEMS
2.4	**7, 10, 11, 13, 14(c)(d), 15, 17**
2.5	**1, 10**
3.1	**1, 9, 12, 14, 38,**
	39(for any nonabelian group)

ADDITIONAL PROBLEMS

1. An equivalence relation θ on a group G is *left invariant* if $a \equiv b \pmod{\theta}$ implies $ca \equiv cb \pmod{\theta}$ for any $a, b, c \in G$, and *right invariant* if $a \equiv b \pmod{\theta}$ implies $ac \equiv bc \pmod{\theta}$ for any $a, b, c \in G$,

(a) Show that θ is a congruence iff it is both left and right invariant.
(b) Show that there is a inclusion-preserving bijection between right invariant equivalence relations on G and subgroups of G given by $\theta \mapsto S_\theta := 1/\theta$ and $S \mapsto \theta_S := \{(x, y) \in G^2 \mid xy^{-1} \in S\}$.
(c) Show that if S is a subgroup, then the cosets of the associated right invariant equivalence relation θ_S are just the right cosets of S (which are sets of the form Sg, $g \in G$).
(d) Show that the function $S \mapsto \theta_S$ is a lattice embedding of the subgroup lattice of G into the lattice of equivalence relations on G, which maps normal subgroups to their associated congruences.

(Results like those in (b), (c) and (d) hold with "right" replaced by "left", "Sg" replaced by "gS", and "θ_S" replaced by "${}_S\theta := \{(x, y) \in G^2 \mid x^{-1}y \in S\}$".)

2. Show that if $S \leq G$ and S has finitely many right cosets in G, then there is a finite set T that is simultaneously a complete set of right coset representatives and a complete set of left coset representatives.

3. Suppose that S is a subgroup of G, θ is a congruence on G, and $N = 1/\theta$ is the associated normal subgroup. Show that $\theta|_S := \theta \cap S^2$ is a congruence on S and that $S \cap N = 1/(\theta|_S)$ is the associated normal subgroup.

Challenge Problem #5 (Prize = 1 coin) It is well known and easy to see that a group satisfying the identity $(xy)^2 = x^2y^2$ is abelian. (The identity says $xyxy = xxyy$, so cancelling yields $yx = xy$ for any x and y.) This problem generalizes it by considering other identities of the form

$$\varepsilon_n\colon\ (xy)^n = x^n y^n.$$

Suppose that G is a finite group satisfying identities $\varepsilon_{n_1}, \ldots, \varepsilon_{n_k}$ for some positive integers $n_1, \ldots, n_k$ satisfying

$$\gcd\left(\binom{n_1}{2}, \ldots, \binom{n_k}{2}\right) = 1.$$

Show that G is abelian.