ALGEBRA 1
HOMEWORK ASSIGNMENT II

(Turn in underlined problems.)

Read Section 1.2–1.3.

SECTION	**PROBLEMS**
1.1	**15, 19, 25, 28**
1.2	**9-13, 15**
1.3	**11, <u>15</u>**

ADDITIONAL PROBLEMS

1. A *congruence* is a compatible equivalence relation.

(a) Show that the kernel of a homomorphism is a congruence.
(b) Show that a congruence is the kernel of a homomorphism.

<u>2.</u> Let θ be a congruence on a group G.

(a) Show that $M := 1/\theta$ is a normal subgroup of G (i.e., a subgroup satisfying $gM = Mg$ for all $g \in G$).
(b) Show that if N is a normal subgroup of G, then $\theta_N := \{(x, y) \in G^2 \mid xy^{-1} \in N\}$ is a congruence on G.
(c) Show that the mappings $\theta \mapsto 1/\theta$ and $N \mapsto \theta_N$ define an inclusion-preserving bijection between the set of congruences on G and the set of normal subgroups of G.
(d) Show that if θ is a congruence on G and $M = 1/\theta$, then the cosets of θ are $\{gM \mid g \in G\} = \{Mg \mid g \in G\}$

<u>3.</u> Given a subset $T \subseteq S_n$ construct a graph G_T by taking $\{1, \ldots, n\}$ to be the vertex set and $\{\{i, j\} \mid (i\ j) \in T\}$ to be the edge set.

(a) Prove that if H is a subgroup of S_n, then the connected components of G_H are complete subgraphs. (Equivalently, any two vertices connected by a path in G_H are connected by a single edge.)

(b) Prove that if $T \subseteq S_n$ is a set of transpositions, then the subgroup of S_n generated by T is all of S_n iff G_T is connected. (Hint for the backward implication: use (a) with H taken to be the subgroup generated by T.)

(c) Conclude that if p is prime, then S_p is generated by any subset of the form $\{\tau, \pi\}$ where τ is a transposition and π is a p-cycle. (Hint: if $\tau = (i\ j)$, then the subgroup generated by $\{\tau, \pi\}$ contains the subset $T \subseteq S_n$ of all permutations of the form $\pi^k \tau \pi^{-k} = (\pi^k(i)\ \pi^k(j))$.)

(d) Show that if n is not prime, then S_n has a subset $\{\tau, \nu\}$ where τ is a transposition, ν is an n-cycle, and $\{\tau, \nu\}$ does not generate S_n.

4. Let $T \subseteq S_n$ be a set of 3-cycles. Define a hypergraph G_T whose vertex set is $\{1, \ldots, n\}$ and whose edge set is $\{\{i, j, k\} \mid (i\ j\ k) \in T\}$. Show that the subgroup of S_n generated by T is A_n iff G_T is connected. (A *hypergraph* is defined like a graph, except edges can have any number of vertices. In this problem, though, all edges have three vertices. A hypergraph is *connected* if any pair of elements a and b are part of a chain $a = u_1, u_2, \ldots, u_n = b$ where any consecutive pair of elements lie in an edge.)

5. Give necessary and sufficient conditions that m and n must satisfy for D_m to be embeddable in S_n.

Challenge Problem! (Prize = 2 Forint!) Prove that there are at least $2^{\lfloor n^2/16 \rfloor}$ different subgroups of S_n.