ALGEBRA 1
HOMEWORK ASSIGNMENT XI

(Turn in underlined problems.)

Read Sections 8.1-8.3, 9.2-9.5.

SECTION	**PROBLEMS**
8.1	**3**
8.2	**4, *6***
8.3	**4, 8**
9.2	***1*, *4*, 6**
9.4	***1, 2, 3*, 9, 17**
9.5	**1, 3**

ADDITIONAL PROBLEMS

(Chinese Remainder Theorem) Suppose that θ_1 and θ_2 are permuting equivalence relations on a set A (i.e., $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$). If $a_1 \equiv a_2 \pmod{\theta_1 \vee \theta_2}$, then the system

$$\begin{aligned} x &\equiv a_1 \pmod{\theta_1} \\ x &\equiv a_2 \pmod{\theta_2} \end{aligned}$$

has a solution in A. Moreover the solution is unique modulo $\theta_1 \cap \theta_2$.

Proof. Since the equivalence relations permute, $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$. Since $(a_1, a_2) \in \theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$ there is an $x \in A$ such that $(a_1, x) \in \theta_1$ and $(x, a_2) \in \theta_2$. This x is a solution. If y were another solution, then $x \equiv a_1 \equiv y \pmod{\theta_1}$ and $x \equiv a_2 \equiv y \pmod{\theta_2}$, so $(x, y) \in \theta_1 \cap \theta_2$. □

1. Write down the Chinese Remainder Theorem for groups and rings with the notion of 'congruence' replacing 'equivalence relation', but use the language of normal subgroups and ideals instead of congruences. Your statement should make use of the fact that any algebra with an underlying group structure has permuting congruences.

2. Let R be a commutative ring with unit.

(a) **Use the Chinese Remainder Theorem (CRT) to show that if $I, J \lhd R$ are *comaximal* ideals, meaning that $I + J = R$, then $R/(I \cap J) \cong R/I \times R/J$.**

(b) **Suppose that $f, g \in k[x]$ are relatively prime polynomials with coefficients in a field k. Show that $k[x]/(f(x)g(x)) \cong k[x]/(f(x)) \times k[x]/(g(x))$.**

(c) **Suppose that $f(x) \in \mathbb{C}[x]$ does not have repeated linear factors. Show that $\mathbb{C}[x]/(f(x)) \cong \mathbb{C}^d$, where $d = \deg(f)$.**

(d) **As a side observation, show that if I and J are comaximal, then $I \cap J = IJ$. (Hint: This part has nothing to do with the CRT. Instead, consider the set of ideals X such that $X(I \cap J) \subseteq IJ$. Show that I and J are in this set, and that the set is closed under join. Then derive the result.)**

3. Verify the following geometric interpretation of the Chinese Remainder Theorem. Let $\mathbf{A}$ be an algebra, and denote the join operation of $\mathrm{Con}(\mathbf{A})$ by $+$ and the least element by 0.

(a) **Show that the function $d\colon A \times A \to \mathrm{Con}(\mathbf{A})$ is a nonarchimedean metric in the sense that**
  (i) $d(a,b) \geq 0$ **with** $d(a,b) = 0$ **iff** $a = b$.
  (ii) $d(a,b) = d(b,a)$
  (iii) $d(a,c) \leq d(a,b) + d(b,c)$

(b) **Define the closed ball of radius θ centered at a to be $B(a,\theta) = \{x \in A \mid d(x,a) \leq \theta\}$. Show that the Chinese Remainder Theorem is the statement: *If two closed balls are near enough to intersect, then they do intersect.* (Here closed balls are "near enough to intersect" if the distance between their centers is less or equal to the sum of their radii.)**

4. (Localization) Let R be a commutative ring and $S \subseteq R$ be a multiplicatively closed subset. Let $\chi\colon R \to S^{-1}R$ be the canonical map.

(a) **Show that an element of R maps to a unit under χ iff it is a divisor of some element of S.**

(b) **Suppose that $\overline{S}$ is the closure of S under divisors. (That is, $ab \in S \Rightarrow a \in \overline{S}$) Show that the natural map $S^{-1}R \to \overline{S}^{-1}R$ is an isomorphism. (The 'natural map' comes from the universal property of presentations. What this problem reduces to is showing that an element of R that is annihilated by some element of $\overline{S}$ is already annihilated by some element of S.)**

(c) **Call a multiplicatively closed set *saturated* if it is closed under divisors. By the Part (b), we never need to localize at any other kind of**

multiplicatively closed set. Show that a subset $S \subseteq R$ is a saturated multiplicatively closed set iff $R - S$ is the union of a set of prime ideals. (Here we allow $0 \in S$ iff we allow R itself to be a prime ideal.) (Hint: Show first that if $a \in R - S$, then $(a) \subseteq R - S$. Thus $R - S$ is a union of ideals. Now show that the maximal ideals in the union are prime. For the other direction, show that the complement of any union of prime ideals is a saturated multiplicatively closed set.)

(d) If I is an ideal in R, then $S^{-1}I$ is an ideal in $S^{-1}R$. If J is an ideal in $S^{-1}R$, then $\chi^{-1}(J)$ is an ideal in R. This is not a bijective correspondence, but show at least that this correspondence is a bijection from the set of prime ideals of R that are disjoint from S and the set of all prime ideals of $S^{-1}R$. (In particular, an ideal is prime in $S^{-1}R$ iff it has the form $S^{-1}P$ for some prime in R that is disjoint from S.)

***5.* Let R be an integral domain.**

(a) Show that if R is finite or is a finite dimensional k-algebra over a field k, then R is a field. (Hint: consider the injectivity/surjectivity of the function $\lambda_r(x) = rx$ for $r \in R - \{0\}$.)

(b) A commutative ring is *zero dimensional* if every prime ideal is maximal. Show that every finite commutative ring or finite dimensional k-algebra is zero dimensional.