

ALGEBRA 1

HOMEWORK ASSIGNMENT I

(Turn in underlined problems.)

Read Chapter 0 and Section 1.1.

PROBLEMS

1. Let X be a set, and let $\text{Inject}(X) = \langle \{f \in X^X \mid f \text{ is an injection}\}; \circ, \text{id} \rangle$. Algebraize the theory of injective functions by showing that

- (a) $\text{Inject}(X)$ satisfies
 - (i) $(f \circ (g \circ h)) = ((f \circ g) \circ h)$,
 - (ii) $\text{id} \circ f = f \circ \text{id} = f$, and
 - (iii) $(f \circ g = f \circ h) \Rightarrow (g = h)$.(I.e., $\text{Inject}(X)$ is a left cancellative monoid.)
- (b) If $M = \langle M; \cdot, 1 \rangle$ is an algebra satisfying (a)(i)–(iii), then M is embeddable in $\text{Inject}(X)$ for some X .

2. Let $\text{Surject}(X) = \langle \{f \in X^X \mid f \text{ is a surjection}\}; \circ, \text{id} \rangle$. Algebraize the theory of surjective functions by showing that

- (a) $\text{Surject}(X)$ is a right cancellative monoid (i.e., satisfies 1(a)(i)-(ii) and
 - (iii)' $(f \circ h = g \circ h) \Rightarrow (f = g)$.)
- (b) If $M = \langle M; \cdot, 1 \rangle$ is a right cancellative monoid, then M is embeddable in $\text{Surject}(X)$ for some X .

Let $\text{Biject}(X) = \langle \{f \in X^X \mid f \text{ is a bijection}\}; \circ, \text{id} \rangle$. It would be natural to guess that $\text{Biject}(X)$ is algebraized by the class of (left *and* right) cancellative monoids, but this is wrong. Any algebra of the form $\text{Biject}(X)$ is indeed a cancellative monoid, but must satisfy stronger properties that do not follow from cancellativity, such as the one in the next problem.

3. Show that $\text{Biject}(X)$ satisfies

$$(f \circ g = f' \circ g') \& (h \circ g = h' \circ g') \& (h \circ k = h' \circ k') \Rightarrow (f \circ k = f' \circ k').$$

4. Modify $\text{Biject}(X)$ to $\text{Perm}(X) = \langle \{f \in X^X \mid f \text{ is a bijection}\}; \circ, {}^{-1}, \text{id} \rangle$. Here f^{-1} is the inverse of the function f . Show that the theory of bijective functions is algebraized by structures $G = \langle G; \cdot, {}^{-1}, 1 \rangle$ satisfying

- (i) $(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$,
- (ii) $1 \cdot x = x \cdot 1 = x$, and
- (iii) $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

That is, show that $\text{Perm}(X)$ satisfies (i)-(iii), and that any structure of type $\langle 2, 1, 0 \rangle$ satisfying (i)-(iii) is embeddable in $\text{Perm}(X)$ for some X .

5. Let $\langle P; \leq \rangle$ be a set equipped with a partial order. Suppose that any two elements of P have a greatest lower bound, i.e.

$$\forall a, b \exists c((c \leq a) \ \& \ (c \leq b) \ \& \ \forall d(((d \leq a) \ \& \ (d \leq b)) \rightarrow (d \leq c))),$$

is satisfied. Algebraize this situation by defining a binary operation $*$ on P such that $a * b$ equals the greatest lower bound of a and b . Find identities satisfied by $*$, and show your list of identities to be complete by proving a representation theorem. (Your representation theorem should show that any algebra $A = \langle A; * \rangle$ satisfying your identities is embeddable into a partially ordered set in such a way that $a * b$ is the greatest lower bound of a and b with respect to the given order.)

Challenge Problem! (Prize = 1 Peso!) Show that the implication in Problem 3 is not a consequence of the associative laws, the unit laws and cancellativity by exhibiting a cancellative monoid that fails the implication. (Such a cancellative monoid cannot be represented as a monoid of bijections.)