Bipartite Graphs and their Idempotent Polymorphisms

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Definitions

Digraph: a finite structure $\mathbf{G} = (V, E)$ where E is a binary relation on V.

Graph: a digraph (V, E) in which E is symmetric and irreflexive.

Polymorphism of a finite relational structure $\mathbf{A} = (A, \ldots)$:
an operation $f : A^n \to A$ $(n \geq 1)$ which preserves each relation of $\mathbf{A}$.

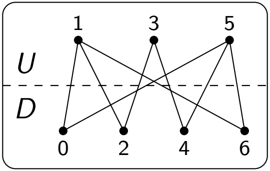
Idempotent operation: an operation f that satisfies $f(x, x, \ldots, x) = x$.

A satisfies a Maltsev condition Σ:
this means that $\mathbf{A}$ has polymorphisms which satisfy the identities in Σ.

Problem

Which finite graphs satisfy your favorite Maltsev condition?

Recall that a graph is bipartite if there exists a partition $V = D \,\dot\cup\, U$ such that all edges are between D and U.



Bipartite graph

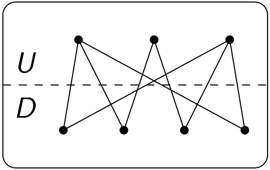
Theorem (Bulatov 2005; Hell, Nešetřil 1990)

Suppose **G** is a graph. If **G** satisfies a nontrivial idempotent Maltsev condition, then **G** is bipartite.

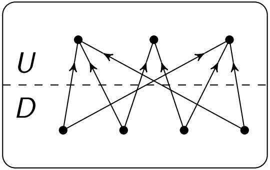
Therefore we restrict our attention to bipartite graphs.

Definition (Larose, Lemaître)

A digraph $\mathbf{G} = (V, E)$ is strongly bipartite if there exists a partition $V = D \cupdot U$ such that $E \subseteq D \times U$.



$\mathbf{G} = (V, E)$
Bipartite graph



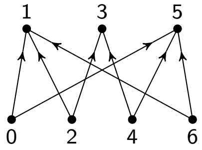
$\vec{\mathbf{G}} = (V, \vec{E})$
Strongly bipartite digraph

Every bipartite graph can be associated with a strongly bipartite digraph, and vice versa.

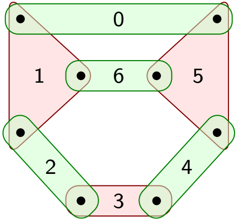
Definition

A 2-equivalence structure is a finite structure $(A; \alpha, \beta)$ where

- α and β are equivalence relations on A.
- $\alpha \cap \beta = 0_A$.



$\vec{\mathbf{G}} = (V, \vec{E})$



A 2-equivalence structure $\mathrm{Eq}(\vec{\mathbf{G}})$

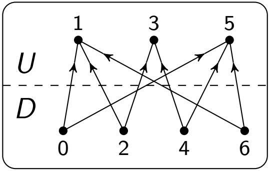
$\alpha =$ [green] blocks
$\beta =$ [red] blocks

Every strongly bipartite digraph can be associated with a 2-equivalence structure, and vice versa.

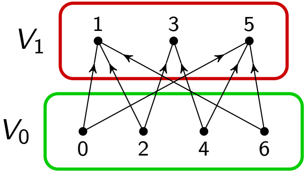
Definition

A 2-sorted digraph is a 2-sorted structure $(V_0, V_1; E)$ where

1. V_0 and V_1 are finite non-empty sets (the universes).
2. $E \subseteq V_0 \times V_1$.



$\vec{\mathbf{G}} = (V, \vec{E})$
Strongly bipartite digraph



$\vec{\mathbf{G}}_{(2)} = (V_0, V_1; \vec{E})$
2-sorted digraph

Every strongly bipartite digraph can be associated with a 2-sorted digraph, and vice versa.

Useful Lemma

Let Σ be an idempotent Maltsev condition such that

1. Every identity in Σ mentions at most two variables;
2. The 2-element graph satisfies Σ.

Let $\mathbf{G}$ be a connected bipartite graph and let $\vec{\mathbf{G}}$, $\mathrm{Eq}(\vec{\mathbf{G}})$, and $\vec{\mathbf{G}}_{(2)}$ be the corresponding strongly bipartite digraph, 2-equivalence structure and 2-sorted digraph respectively.

If any of $\mathbf{G}$, $\vec{\mathbf{G}}$, $\mathrm{Eq}(\vec{\mathbf{G}})$ or $\vec{\mathbf{G}}_{(2)}$ satisfy Σ, then all satisfy Σ.

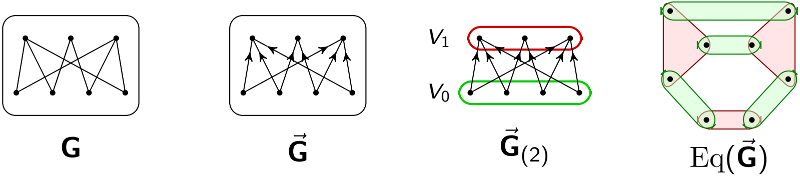
Remark. By an n-ary polymorphism of $\vec{\mathbf{G}}_{(2)} = (V_0, V_1; E)$ I mean a pair $\mathbf{f} = (f_0, f_1)$ where $f_i : (V_i)^n \to V_i$ and such that f_0, f_1 jointly preserve E:

$$\text{if } (a_1, b_1), \dots, (a_n, b_n) \in E \text{ then } (f_0(\mathbf{a}), f_1(\mathbf{b})) \in E.$$

Lemma (summary)

- Σ an idempotent Maltsev condition satisfying two hypotheses.
- G connected, bipartite.

If any of $\mathbf{G}$, $\vec{\mathbf{G}}$, $\vec{\mathbf{G}}_{(2)}$ or $\mathrm{Eq}(\vec{\mathbf{G}})$ satisfy Σ, then all satisfy Σ.



Proof idea

Pp-interpretations: $\mathrm{Eq}(\vec{\mathbf{G}}) \equiv_{pp} \vec{\mathbf{G}}_{(2)} \leq_{pp} \vec{\mathbf{G}} \leq_{pp} \mathbf{G}^c$.

Thus it suffices to show that $\vec{\mathbf{G}}_{(2)} \models \Sigma \;\Rightarrow\; \mathbf{G} \models \Sigma$.

There is a recipe for doing this.

Question (Benoit Larose, Nov' 2012)

Does there exist a bipartite graph which:

1. Satisfies the Maltsev condition for congruence n-permutability (n-PERM) for some n, and
2. Satisfies the Maltsev condition for congruence meet-semidistributivity (SD($\wedge$)), but
3. Does NOT have a near-unanimity (NU) polymorphism.

Theorem (W)

If a bipartite graph is n-PERM for some $n \leq 5$, then it is NU.

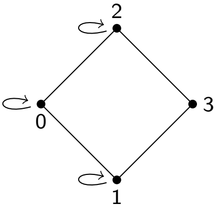
Proof idea

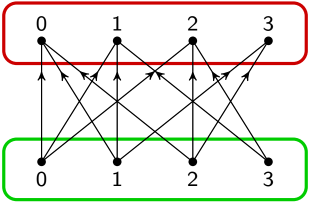
Analyze 2-sorted digraphs. Characterize which are n-permutable for $n \leq 5$.

The previous result does not extend to 6-PERM.

Example

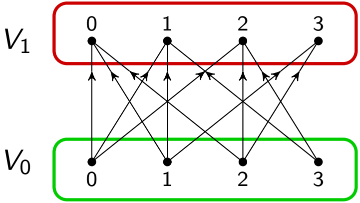
There exists a bipartite graph which is 6-PERM and SD($\wedge$), but does not have an NU polymorphism.





All the structures on this page are 6-PERM and SD($\wedge$) but have no NU.

No NU



Suppose $\mathbf{f} = (f_0, f_1)$ is an n-ary NU polymorphism.

$\{3\}$ is absorbing for each f_i.

Therefore $\{1,2\}$ is absorbing for each f_i. Consider

$$\begin{aligned} f_1(0,2,2,2,\ldots,2) &= 2 \\ f_0(1,0,2,2,\ldots,2) &= 2 \end{aligned}$$

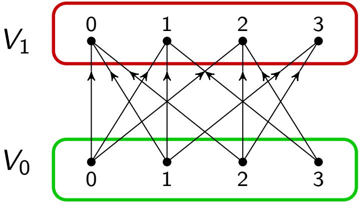
Bottom line must be in $\{1,2\}$ (absorbing), and connected to 2, so is 2.

Similarly, show

$$\begin{aligned} f(1,1,2,2,\ldots,2) &= 2 \\ f(1,1,1,2,\ldots,2) &= 2 \end{aligned}$$

etc.

5-PERM



First, delete both 3's.

The resulting subgraph has 3-PERM polymorphisms $\mathbf{p}^1 = (p_0^1, p_1^1)$, $\mathbf{p}^2 = (p_0^2, q_1^2)$ such that all p_j^i preserve $\{1,2\}$.

Lemma

Suppose $\mathbf{G} = (V_0, V_1; E)$ is a 2-sorted digraph, $\mathbf{H} = (H_0, H_1; E')$ is a retract of $\mathbf{G}$, and $\mathbf{r} = (r_0, r_1)$ is a **strong retraction** of $\mathbf{G}$ onto $\mathbf{H}$, i.e.,

- $N(a) \subseteq N(r_0(a))$ for all $a \in V_0$, and dually.

Suppose $\mathbf{H}$ has n-PERM polymorphisms $\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^{n-1}$ satisfying

- For all $a \in V_0 \setminus H_0$, all p_1^i preserve $N(a) \cap H_1$, and dually.

Then $\mathbf{G}$ has $(n+2)$-PERM polymorphisms.

Apply the Lemma with both 3's being sent to 0.

Problems

1. Characterize the 6-PERM bipartite graphs.
2. Characterize the bipartite graphs which are n-PERM for some n.
3. (Larose) Prove that if a bipartite graph $\mathbf{G}$ is 6-PERM (or n-PERM) and SD($\wedge$), then $\mathrm{CSP}(\mathbf{G}^c)$ is in $\textsc{LogSpace}$.

Thank you!