Congruence n-permutable varieties

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Interpretability, Maltsev Conditions

Definition (W. Neumann, W. Taylor)

Let $\mathcal{U}$ and $\mathcal{V}$ be varieties.

- (Kearnes-Kiss): "The condition that $\mathcal{U} \leq \mathcal{V}$ may be expressed as there is a set of $\mathcal{V}$-terms corresponding to the basic operation symbols of $\mathcal{U}$ such that the set of identities corresponding to the axioms of $\mathcal{U}$ hold in $\mathcal{V}$."
- Suppose that the operation symbols of $\mathcal{U}$ are $\{f_i : i \in I\}$. We say that $\mathcal{U}$ is interpretable in $\mathcal{V}$, and write $\mathcal{U} \leq \mathcal{V}$, if for every $i \in I$ there is a $\mathcal{V}$-term t_i of the same arity as f_i such that for all $\mathbf{A} \in \mathcal{V}$, the algebra $\langle A, t_i^{\mathbf{A}}(i \in I)\rangle$ is a member of $\mathcal{U}$.
- If $\mathcal{U}$ is a finitely presented variety, then the condition $\mathcal{U} \leq \mathcal{V}$ on $\mathcal{V}$ is called the strong Maltsev condition defined by $\mathcal{U}$, and $\{\mathcal{V} : \mathcal{U} \leq \mathcal{V}\}$ is the strong Maltsev class defined by this condition.

A very weak Maltsev condition

Theorem (Taylor, Hobby-McKenzie)

Let $\mathcal{V}$ be a locally finite idempotent variety. The following are equivalent:

- $\mathcal{V} \not\leq$ *Sets,*
- $\mathcal{V}$ *has a Taylor term,*
- $\mathcal{V}$ *satisfies the congruence inclusion* $\alpha \cap (\beta \circ \gamma) \subseteq \gamma_m \circ \beta_m$*,*
- $\mathcal{V}$ *omits the unary type.*

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- $\mathcal{V}$ *satisfies the congruence inclusion* $\alpha \cap (\beta \circ_4 \gamma) \subseteq w(\alpha, \beta, \gamma)$*, for some lattice word* w *with* $w(p, q, r) < p \wedge (q \vee r)$*,*
- $\mathcal{V}$ *has no member with a nonzero strongly abelian congruence.*

Avoiding Semilattices

Theorem (Hobby-McKenzie)

Let $\mathcal{V}$ be a locally finite idempotent variety. The following are equivalent:

- $\mathcal{V} \not\leq$ *Semilattices,*
- $\mathcal{V}$ *has a Hobby-McKenzie term,*
- $\mathcal{V}$ *satisfies* $\alpha \cap (\beta \circ \gamma) \subseteq (\gamma \vee (\alpha \wedge \beta)) \circ (\beta \vee (\alpha \wedge \gamma))$,
- $\mathcal{V}$ *omits the unary and semilattice types.*

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- $\mathcal{V}$ *satisfies* $\alpha \cap (\beta \circ \gamma) \subseteq (\gamma \vee (\alpha \wedge \beta)) \circ (\beta \vee (\alpha \wedge \gamma))$,
- $\mathcal{V}$ *satisfies a nontrivial congruence identity,*
- $\mathcal{V}$ *has no member with a nontrivial rectangular congruence.*

Avoiding Modules

Theorem (Hobby-McKenzie)

Let $\mathcal{V}$ be a locally finite idempotent variety. The following are equivalent:

- *$\mathcal{V} \not\leq \mathcal{M}$ for every non-trivial locally finite variety of modules,*
- *$\mathcal{V}$ is congruence meet semidistributive,*
- *$\mathcal{V}$ satisfies the congruence inclusion $\alpha \cap (\beta \circ \gamma) \subseteq \beta_m \cap \gamma_m$, for some m,*
- *$\mathcal{V}$ omits the unary and affine types.*

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- *$\mathcal{V}$ satisfies the congruence inclusion $\alpha \cap (\beta \circ \gamma) \subseteq \beta_m$, for some m,*
- *$\mathcal{V}$ has no member with a nontrivial abelian congruence.*

Congruence n-permutability

Definition

A variety $\mathcal{V}$ is congruence n-permutable if for all $\mathbf{A} \in \mathcal{V}$ and α, $\beta \in \mathrm{Con}\,(\mathbf{A})$, $\alpha \circ_n \beta = \beta \circ_n \alpha$.

Theorem (Hagemann-Mitschke)

The following are equivalent for a variety $\mathcal{V}$ and $n > 1$:

- *$\mathcal{V}$ is congruence n-permutable,*
- *$\mathcal{V}$ has terms $p_i(x, y, z)$, for $1 \leq i < n$, which satisfy the identities:*

$$\begin{aligned} x &= p_1(x, y, y) \\ p_i(x, x, y) &= p_{i+1}(x, y, y) \quad \text{for } 1 \leq i < n-1 \\ p_{n-1}(x, x, y) &= y. \end{aligned}$$

Avoiding distributive lattices

Theorem (Hobby-McKenzie)

Let $\mathcal{V}$ be a locally finite idempotent variety. The following are equivalent:

- $\mathcal{V} \not\leq$ *Dist, the variety of distributive lattices,*
- $\mathcal{V}$ *has Hagemann-Mitschke terms,*
- $\mathcal{V}$ *is congruence n-permutable for some* $n > 1$,
- $\mathcal{V}$ *omits the unary, lattice, and semilattice types.*

Orderable algebras

Definition

An algebra $\mathbf{A}$ is orderable if there is a compatible non-trivial partial order on A. This is equivalent to all of the term operations of $\mathbf{A}$ being monotone with respect to some non-trivial partial order on A.

Theorem (Hagemann)

If a variety $\mathcal{V}$ contains an orderable algebra, then $\mathcal{V}$ is not congruence n-permutable for any $n > 1$.

Proof.

Suppose that $\leq$ is a compatible non-trivial order on some $\mathbf{A} \in \mathcal{V}$ and let $a < b$ in A. Using the Hagemann-Mitschke terms for congruence n-permutability yields the following contradiction:

$$b = p_1(b, a, a) \leq p_1(b, b, a) = p_2(b, a, a) \leq \cdots = p_n(b, a, a) = a.$$

□

A converse

Theorem (Hagemann)

A variety $\mathcal{V}$ does not contain an orderable algebra if and only if it is congruence n-permutable for some $n > 1$.

Proof.

- Construct an algebra in $\mathcal{V}$ that has a compatible order and show that the triviality of this order gives rise to Hagemann-Mitschke terms.
- Let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ generated by $\mathbf{x}$ and $\mathbf{y}$.
- Let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^2$ generated by $\{(\mathbf{x}, \mathbf{x}), (\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{y})\}$ and
- let $\preceq$ be the transitive closure of R, a compatible quasi-order on $\mathbf{F}^2$.
- $\preceq$ is trivial, and so $\mathbf{y} \preceq \mathbf{x}$.
- From this we get terms for n-permutability for some $n > 1$.

□

A better converse

Theorem (Valeriote-Willard)

A idempotent variety $\mathcal{V}$ is congruence n-permutable for some $n > 1$ if and only if $\mathcal{V} \not\leq Dist$.

Remarks

- *One direction has already been established, since Hagemann-Mitschke terms can't be modelled by any orderable algebra.*
- *The converse amounts to showing that $\mathcal{V}$ must contain a 2-element orderable algebra, if it contains some orderable algebra.*

A better converse

Lemma

Let $\mathcal{V}$ be an idempotent variety that contains an orderable algebra. Then $\mathcal{V}$ contains a 2-element orderable algebra.

Remarks

- *Complications arise when $\leq$ is a dense order of P.*
- *Otherwise, if P is finite, or more generally, if there is a covering pair $a < b$ in P then a 2-element orderable algebra can easily be constructed as a subalgebra of $\mathbf{P}$.*
- *If b covers a in P then $\{a, b\} = a\uparrow \cap b\downarrow$ is the intersection of two subalgebras of $\mathbf{P}$ and so is also a subalgebra.*

Ultrapower Lemma

Lemma

If **P** *is an idempotent orderable algebra whose ordering* $\leq$ *is dense, then some subalgebra of an ultrapower of* **P** *has a 2-element quotient (and so any variety containing* **P** *has a 2-element orderable algebra).*

Remarks

- *So,* $HSP_U(\mathbf{P})$ *contains a 2-element orderable algebra.*
- *We may assume that* **P** *is bounded by elements* $0 < 1$*.*
- *Any ultrapower of an orderable algebra is orderable and will belong to the variety generated by the algebra.*

An ultrafilter on P

Definition

- For $A \subseteq P$, let $N_A = (A\uparrow) \setminus (A\downarrow)$.
- Let $\mathcal{J} = \{N_A : A \subseteq (P \setminus \{0\})\} \cup \{\{0\}\}$.

Example

When P is the unit interval $[0,1]$ and $\leq$ is the usual ordering, then $\mathcal{J}$ consists of all intervals of the form $(\epsilon, 1]$ and $[\epsilon, 1]$ for $\epsilon > 0$, and $\{0\}$.

Claim

P is not equal to the union of any finite subset from $\mathcal{J}$ and so there is an ultrafilter $\mathcal{U}$ on P that is disjoint from $\mathcal{J}$.

Claim

If $Z \in \mathcal{U}$ then Z is "downward closed" in P, i.e., there is some $0 < x \in Z$ such that for all $p \in P$ with $0 < p < x$, there is $u_x \in Z$ with $u_x \leq p$.

An ultrapower of **P**

Definition

- Let $\mathbf{U}$ be the ultrapower $\mathbf{P}^P/\mathcal{U}$.
- Let $\mathbf{0}$ and $\mathbf{1}$ be the images of the constant 0 function and the identity function on P in $\mathbf{U}$.
- Let $\mathbf{S}$ be the subalgebra of $\mathbf{U}$ generated by $\{\mathbf{0}, \mathbf{1}\}$.

Claim

- *(Key Algebraic fact): For all terms $t(x_1, \ldots, x_n)$ and all $b_i > \mathbf{0}$ in $\mathbf{S}$, $t(\mathbf{0}, b_2, \ldots, b_n) > \mathbf{0}$ if and only if $t(\mathbf{0}, \mathbf{1}, \ldots, \mathbf{1}) > \mathbf{0}$.*
- $\mathbf{S}$ *has a 2-element quotient via the congruence that identifies all elements of S that are not equal to $\mathbf{0}$.*