Possible classification of varieties modelable on finite simplicial complexes.

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The original question.

A. D. Wallace defined the inquiry in 1955, when he asked, "Which spaces admit what structures?"

Here "structure," means **the existence of continuous operations identically satisfying certain equations**: e.g., the structure of a topological group or a topological lattice, and so on.

Here we survey the current state of knowledge in this area, espcecially **for finite simplicial complexes,** and ask some refined versions of Wallace's question.

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Each of these may be realized on a finite simplicial complex.

interpretability

For equational theories Γ and Δ, we say that Γ is interpretable in Δ, written $\Gamma \leq \Delta$, iff there exist terms γ_t in the Δ-language such that, for each algebra $\mathbf{D} \in \Delta$, the algebra $(D, \overline{\gamma}_t)_{t \in T}$ is an algebra of Γ.

Example: Γ is Abelian groups of exponent 2, Δ is Boolean algebra, and γ_+ is symmetric difference. (Well known.)

Obviously, if $A \models \Delta$ and $\Gamma \leq \Delta$, then $A \models \Gamma$.

Therefore it is important to know $A \models \Delta$ for Δ as high as possible, and to know $A \not\models \Gamma$ for Γ as low as possible.

Example and open question

Let Λ_n $(n = 1, 2, \ldots)$ have axioms for distributive lattice theory, plus the following:

$$a_1 \wedge a_2 \approx a_1, \quad a_2 \wedge a_3 \approx a_2, \quad \cdots, \quad a_{n-1} \wedge a_n \approx a_{n-1}$$
$$f(0) \approx 0, \quad f(a_1) \approx 1, \quad f(a_2) \approx 0, \quad f(a_3) \approx 1, \quad \cdots$$
$$f(1) \approx 1 \text{ if } n \text{ is even}, \; 0 \text{ otherwise.}$$

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Even their join (disjoint union) is compatible with an interval I. Is this a maximal theory compatible with I?

We have not identified **any** maximal theory compatible with I.

Today's central question

The spaces A associated to finite simplicial complexes are also known as **finitely triangulable**. We may also say A is a **finite space**. They seem simple enough, but much of the chaotic behavior of "$\models$" occurs already in the finite realm. We let

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Question: Does there exist a recursive sequence $\Sigma_0, \Sigma_1 \ldots$ (with each Σ_n a finite set of equations) such that $\Sigma \in J$ if and only if for some n, $\Sigma \leq \Sigma_n$ in the interpretability lattice? If yes, please be more specific.

Questions surrounding the central question.

To repeat: we consider the possibility of finding finite theories Σ_n such that:

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We are further interested in such things as: the **arities** that might be required for such generators Σ_n; the **simplicity** of operations needed to model the Σ_n; and whether the known examples more or less comprise the totality of Σ_n that will be required.

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Corollary J contains group theory (using $A = S^1$) and semilattice theory (using $A = I$), but not their join.

Thus J is not an ideal.

What operations are needed to show that $\Sigma \in J$?

For each $\Sigma \in J$, do there exist a finite complex A and continuous piecewise multilinear operations $\overline{F}_t$ on A such that $(A, \overline{F}_t)_{t \in T} \models \Sigma$?

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If not, does there exist some reasonable enlargement of the category "piecewise multilinear" for which the answer is yes?

For example, in the previously described theory Λ_n, we could satisfy the equations on $I = [-1, 1]$ with (fancy!) Chebysheff polynomials, but in fact Λ_n can also be satisfied with piecewise linear maps. (See next slide.)

Reprise of the theory Λ_n.

$$a_1 \wedge a_2 \approx a_1, \quad a_2 \wedge a_3 \approx a_2, \quad \cdots, \quad a_{n-1} \wedge a_n \approx a_{n-1}$$

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One could use a fancy polynomial to make a function $\overline{f}$ going back and forth between the endpoints of the interval. In fact one can do it more simply by making $\overline{f}$ a piecewise-linear function (of one variable).

In all examples that we understand in detail, piecewise multilinear functions seem to do the job. Why?

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And of course, if both answers are no, then we could ask a similar question for every arity.

Perspective on our questions.

Any system for algebraic computation, if it is to be both **infinite** and **practical**, requires some workable approximation to the finite realm. Two ways of making such approximation available are

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- recursiveness (e.g. as seen for rational numbers), and
- topological approximation (e.g. as for reals).

In the latter realm, practicality further demands some easily described spaces, such as finite simplicial complexes.

We conclude this brief report with a brief **catalog of known examples of theories modeled on finite spaces.** Obviously the desired theories Σ_n will have to account for all these examples.

Can the list be made complete?

Known examples 1.

Distributive lattices with 0, 1

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Any other group varieties?

Any compact group could play a role here. Which of them satisfy identities that need to be included?

Known examples 2.

Any consistent set of simple equations.

E.g. $2/3$ minority.

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Power Varieties.

For any theory Σ, and for any $n = 2, 3, \ldots$, there is a theory $\Sigma^{[n]}$ each of whose (topological) models is the n-th power of a (topological) model of Σ (with a small amount of further structure).

J is closed under the formation of $\Sigma^{[n]}$ from Σ, for every n.

Known examples 3.

A few isolated(?) theories.

One-one not onto:

$$F(x,y,0) \approx x, \quad F(x,y,1) \approx y,$$
$$\psi(\theta(x)) \approx x, \quad \phi(\theta(x)) \approx 0, \quad \phi(1) \approx 1.$$

Possibly some entropic equations:

$$F(x,x) \approx x, \quad F(F(x,y),F(u,v)) \approx F(F(x,u),F(y,v)).$$

A certain Σ rules out all spaces with the fixed-point property. Does it rule out all finite spaces?

$$F(x,u,v) \approx u; \qquad F(\phi(x),u,v) \approx v.$$

Recapitulation of question.

In the three previous slides, have we come close to including all theories modelable on finite spaces? How about all known examples of such theories?

Further questions.

For a fixed finite space A, we could modify the previous questions, replacing J by J_A, the class of theories that are modelable on A. (And thus

$$J = \bigcup_{\text{all } A} J_A \,.)$$

Here each J_A **is** an ideal in the interpretability lattice, but J_A is not closed under the formation of $\Sigma^{[n]}$. All the questions we have asked for J remain open for J_A, except for a few special A. In particular, they remain open for $A = I$, an interval.

references

The article (40 pages):

http://math.colorado.edu/∼wtaylor/classify.pdf

This talk (39 clickstops):

http://math.colorado.edu/∼wtaylor/classbeamer.pdf