Likelihood a finite lattice has an intransitive G-Set representation

Speaker: Steve Seif

Abstract: Results on the number of finite lattices that can be represented by an intransitive G-Set are presented.

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- A lattice L is **represented** by an M-Set $\mathbf{X} = \langle X, M \rangle$ if $L \cong Con(\mathbf{X})$.
- The lattice L is *intranstive G-Set representable* if it is represented by an intransitive G-Set; otherwise, L is *G-Set-transitivity-forcing* .

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- If answer to *FLRP* is "no", is there a finite lattice that is finitely representable but is not representable by a (transitive) G-Set? In other words, do finite groups have "dominion" over finite lattice representations?
- Related to the last question, but independent of *FLRP*: (How) can the finite lattices that are intransitively-G-Set representable be described?

Intransitive G-Set (with k orbits) will be presented as follows:

$$\mathbf{Y} = \langle \sqcup_{i\in[k]} X_i, G \rangle$$

where for $i \in [k] = \{1, \ldots, k\}$, X_i is an orbit of $\mathbf{Y}$.

G^k acts on $\sqcup_{i\in[k]} X_i$: $(g_1, \ldots, g_k)(x_i) = g_i(x_i)$, where the outcome of $g_i(x_i)$ is determined by $\langle X_i, G \rangle$. Let

$$\mathbf{Y}^* = \langle \sqcup_{i\in[k]} X_i, G^k \rangle$$

. Note that if $x_i \in X_i, x_j \in X_j, i \neq j$, that G acts transitively on orbits X_i and X_j implies that $Cg(x_i, x_j)$ in $\mathbf{Y}^*$ has one non-singleton class, $X_i \cup X_j$.

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$$(\alpha_1, \ldots, \alpha_k, \beta)$$

where $\alpha_i \in Con(\mathbf{X}_i)$ and $\beta \in \Pi(k)$ whose influence is: If $(i, j) \in \beta$, then both α_1, α_2 are universal congruences.

Π-product lattices

From the last slide: Congruences of $\mathbf{Y}* = \langle \sqcup_{i\in[k]} X_i, G^k \rangle$ can be described by $\{(\alpha_1, \ldots, \alpha_k, \beta)$ tuples, where $(i,j) \in \beta$ implies α_i, α_j are universal on X_i, X_j resp.

Def'n. Let $L_1, \ldots, L_k$ be a multiset of lattices, and $\Pi(L_1, \ldots, L_k)$, a Π-product lattice, is:

$\{(a_1, \ldots, a_k, \beta) : a_i \in L_i,\ \beta \in \Pi(k), \text{and } (i,j) \in \beta \text{ implies} \quad \text{that } a_i = 1_i \text{ and } a_j = 1_j\}$.

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Lastly, the trivial lattice is defined to be a Π-product lattice.

Note that $Con(\mathbf{Y}^*)$ (which turns out to be a 0,1 cover-preserving sublattice of $Con(\mathbf{Y})$) is the Π-product lattice $\Pi(Con(\mathbf{X}_1), \ldots, Con(\mathbf{X}_k))$.

Elementary properties of Π-product lattices

Lemma A: 1. Every lattice having a single co-atom is isomorphic to a Π-product, with factors that are not uniquely determined.

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Lemma B: *Every* Π-product lattice $\Pi(L_1, \ldots, L_k)$ with algebraic factors has a representation as an intransitive G-Set having k orbits, orbits with congruence lattices isom. to $L_1, \ldots, L_k$.

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$\mathbf{Y}^*$ satisfies the following property, *Property K*:

If $x_i \in X_i$ and $x_j \in X_j$ are in different orbits, then
$X_i \times X_j \subset Cg(x_i, x_j)$.

(which further implies $Cg(x_i, x_j)$ has one non-singleton orbit,
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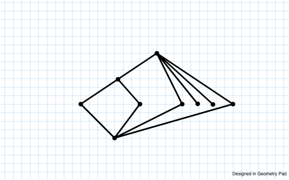
The forward direction of Proposition 1 below is implicit from the discussion on last slide; the other direction is more interesting.

Proposition 1. An intransitive G-Set $\mathbf{Y}$ satisfies Property K iff $Con(\mathbf{Y})$ is isomorphic to a Π-product lattice.

A subclass whose intransitive G-Set representable lattices can be completely described

Defn. 1. For $n \in \mathbb{N}$, $A(n)$ is the lattice that looks like this: $A(4)$ is below, having 4 co-atoms that are also atoms).



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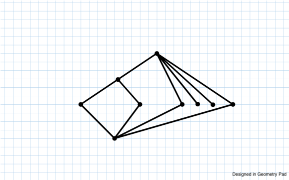

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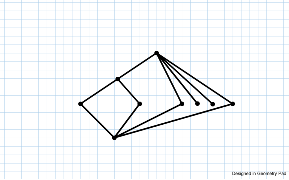

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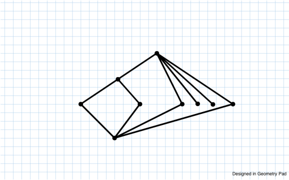

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Corollary A finite lattice L in $\mathcal{N}$ is either a Π-product lattice, in which it has an intransitive G-Set representation, or L is G-Set-transitivity-forcing.

Arbitrary algebraic lattices: $\mathbf{\Pi}(L)$

Theorem 2. Let L be any algebraic lattice. There exists a sublattice of L, $\mathbf{\Pi}(L)$, a certain 0,1 cover-preserving sublattice isomorphic to a Π-product lattice $\Pi(\{L_i : i \in I\})$ such that

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3. Moreover, if $|I| > 2$, $L_i \cong Con(\mathbf{X}_{\phi(i)})$, all $i \in I$.

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Back to finite-orbit intrans G-Sets

Let $l(n)$ be the number of isomorphism classes of n-lattices.
Fact: $log(l(n)) \in \Omega(n^{3/2})$. (Attribute)
An analysis of the congruence lattices of two-orbit G-Sets yields the following useful proposition.

Proposition Given two non-trivial finite lattice L_1, L_2, and a pos. int. n, **there exist no more than** n^n lattices L having a two-orbit G-Set $\mathbf{Z} = \langle X_1 \sqcup X_2, G \rangle$ satisfying $Con(\mathbf{Z}) \cong L$, $|Con(\mathbf{Z})| = n$, and $Con(X_1) \cong L_1, Con(X_2) \cong L_2$.
Corollary There are no more than $n^{n+2} l(\lceil n/2 \rceil)$ n-lattices that are intran. G-Set representable.

Since finite lattices are closed under ordinal sum, and thus $l(2n+1) \geq l(n)l(n+1)$, $\frac{l(\lceil n/2 \rceil)}{l(n)}$ is in the vicinity of $l(\lceil n/2 \rceil)$, a very large number that dominates n^{n+2}.

Asympotic dichotomy for finite lattices

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In fact, there exists $k > 0$ such that for all n high enough, a lattice chosen randomly from among the non-Π-product n-lattices has less than $\frac{1}{2^{kn^{3/2}}}$ likelihood of having an intrans G-Set representation.
Outside of Π*-product lattices, there really aren't any intransitive G-Set representable lattices....*

Specializing to subclasses of finite lattices

Any class C of finite lattices

1. that is closed under ordinal sums,
2. contains $\mathbf{2} \times \mathbf{2}$, and
3. for which there exists $k > 1$ such that for n high enough, $log(l_C(n)) \geq n^k$

satisfies the same "asymptotic dichotomy" as described in the last theorem, Theorem 3 above.

Theorem 4. For a class C satisfying 1.-3. above, there exists $k > 0$ such that for all n high enough, a randomly chosen C lattice from among non-Π-product n-lattices has a representation as an intransitive G-Set with likelihood less $\frac{1}{l_C(kn)}$, where $l_C(n)$ is the number of isomorphism classes of n-lattices in C.

Question (Maybe this is known): Do all varieties properly containing the distributive lattices satisfy 3. above?

Questions

The same asymptotic dichotomy also *seems* to hold if one is restricted to the subclass of finite lattices that are finitely represented, but one has to change from "G-Sets" to so-called "flat M-Sets" those M-Sets that are a "sum" of transitive M-Sets. That is, *among only lattices that are finitely representable*, with high likelihood, a finite lattice is either a Π-product lattice or is flat-transitivity-forcing.

Defn. 1. A finite lattice L is *(finitely)-transitivity-forcing* if all of its (finite) representations are transitive.

2. Let $t(n)$ be the number of isom. classes of finitely-transitivity-forcing lattices.
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Question: Averaged over isomorphism classes, is the average number of atoms of an n-lattice $\Theta(n^{1/2})$?