Distributive integral residuated lattices have the FEP

Nikolaos Galatos
University of Denver
ngalatos@du.edu

April, 2013

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

FEP

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ is finitely axiomatizable, then it's universal theory is decidable.

FEP

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ is finitely axiomatizable, then it's universal theory is decidable.

Fact. Varieties with FEP are generated as quasivarieties by their finite members.

Fact. If $\mathcal{K}$ forms the algebraic semantics of a logical system $\vdash$, then the latter has the *strong finite model property*:
if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model.

FEP

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ is finitely axiomatizable, then it's universal theory is decidable.

Fact. Varieties with FEP are generated as quasivarieties by their finite members.

Fact. If $\mathcal{K}$ forms the algebraic semantics of a logical system $\vdash$, then the latter has the *strong finite model property*:
if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model.

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

FEP for DIRL

DIRL: the variety of distributive, integral $(x \leq 1)$ residuated lattices.

FEP for DIRL

DIRL: the variety of distributive, integral ($x \leq 1$) residuated lattices.

Examples of DIRLs include:

- Boolean algebras (classical logic)
- Heyting algebras (intuitionistic logic)
- MV-algebras (many-valued logic)
- BL-algebras
- negative cones of lattice-ordered groups
- ideals of Prüfer domains

FEP for DIRL

DIRL: the variety of distributive, integral ($x \leq 1$) residuated lattices.

Examples of DIRLs include:

- Boolean algebras (classical logic)
- Heyting algebras (intuitionistic logic)
- MV-algebras (many-valued logic)
- BL-algebras
- negative cones of lattice-ordered groups
- ideals of Prüfer domains

Theorem. Every subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$ has the FEP.

The plan

Let $\mathcal{V}$ be a subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$. To establish the FEP for $\mathcal{V}$, for every $\mathbf{A}$ in $\mathcal{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, we construct an algebra $\mathbf{D}$ such that

- $\mathbf{D} \in \mathcal{V}$
- $\mathbf{B}$ embeds in $\mathbf{D}$
- $\mathbf{D}$ is finite

The plan

Let $\mathcal{V}$ be a subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$. To establish the FEP for $\mathcal{V}$, for every $\mathbf{A}$ in $\mathcal{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, we construct an algebra $\mathbf{D}$ such that

- $\mathbf{D} \in \mathcal{V}$
- $\mathbf{B}$ embeds in $\mathbf{D}$
- $\mathbf{D}$ is finite

The corresponding result for subvarieties of IRL axiomatized over $\{\vee, \cdot, 1\}$ is contained in

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, Transactions of the AMS.

and it is essentially based on Dedekind-MacNeille completions.

The plan

Let $\mathcal{V}$ be a subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$. To establish the FEP for $\mathcal{V}$, for every $\mathbf{A}$ in $\mathcal{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, we construct an algebra $\mathbf{D}$ such that

- $\mathbf{D} \in \mathcal{V}$
- $\mathbf{B}$ embeds in $\mathbf{D}$
- $\mathbf{D}$ is finite

The corresponding result for subvarieties of IRL axiomatized over $\{\vee, \cdot, 1\}$ is contained in

N. Galatos and P. Jipsen. Residuated frames and applications to decidability, Transactions of the AMS.

and it is essentially based on Dedekind-MacNeille completions. The latter do not preserve distributivity so we use a distributive version of the Dedekind-MacNeille completion defined in

N. Galatos and P. Jipsen. Cut elimination and the finite model property for distributive FL, manuscript.

Galois algebra

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \textcircled{\wedge}, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W.

Galois algebra

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \bigcirc\!\!\!\!\wedge, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W. To specify these subsets we define $W' = S_W \times B$, where S_W contains all unary linear polynomials (aka sections) over $(W, \circ, \bigcirc\!\!\!\!\wedge, 1)$. Also we define and

$$x \sqsubseteq (u, b) \text{ iff } u(x) \leq_{\mathbf{A}} b.$$

Galois algebra

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \textcircled{\wedge}, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W. To specify these subsets we define $W' = S_W \times B$, where S_W contains all unary linear polynomials (aka sections) over $(W, \circ, \textcircled{\wedge}, 1)$. Also we define and

$$x \sqsubseteq (u, b) \text{ iff } u(x) \leq_{\mathbf{A}} b.$$

Then $\mathbf{W} = (W, W', \sqsubseteq)$ is an example of a *lattice frame*. (Dedekind, McNeille, Birkhoff) These play the role of *Kripke frames* for non-distributive logics. We have two set of worlds: W for the join-irreducibles and W' for the meet-irreducibles.

Galois algebra

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \bigovert\!\!\wedge, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W. To specify these subsets we define $W' = S_W \times B$, where S_W contains all unary linear polynomials (aka sections) over $(W, \circ, \bigcirc\!\!\!\!\wedge, 1)$. Also we define and

$$x \sqsubseteq (u, b) \text{ iff } u(x) \leq_{\mathbf{A}} b.$$

Then $\mathbf{W} = (W, W', \sqsubseteq)$ is an example of a *lattice frame*. (Dedekind, McNeille, Birkhoff) These play the role of *Kripke frames* for non-distributive logics. We have two set of worlds: W for the join-irreducibles and W' for the meet-irreducibles.

The *Galois algebra* of $\mathbf{W}$ is $\mathbf{W}^+ = (\mathcal{P}(W)_{\gamma_\sqsubseteq}, \cap, \cup_{\gamma_\sqsubseteq})$ and it is a complete lattice.

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \owedge, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W. To specify these subsets we define $W' = S_W \times B$, where S_W contains all unary linear polynomials (aka sections) over $(W, \circ, \owedge, 1)$. Also we define and

$$x \sqsubseteq (u, b) \text{ iff } u(x) \leq_{\mathbf{A}} b.$$

Then $\mathbf{W} = (W, W', \sqsubseteq)$ is an example of a *lattice frame*. (Dedekind, McNeille, Birkhoff) These play the role of *Kripke frames* for non-distributive logics. We have two set of worlds: W for the join-irreducibles and W' for the meet-irreducibles.

The *Galois algebra* of $\mathbf{W}$ is $\mathbf{W}^+ = (\mathcal{P}(W)_{\gamma_\sqsubseteq}, \cap, \cup_{\gamma_\sqsubseteq})$ and it is a complete lattice. For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^{\rhd} = \{b \in W' : x \sqsubseteq b, \text{ for all } x \in X\}$$
$$Y^{\lhd} = \{a \in W : a \sqsubseteq y, \text{ for all } y \in Y\}$$

Consider the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, which we denote by $(W, \circ, \bigcirc\!\!\!\!\wedge, 1)$; this is possibly infinite. Then $\mathbf{D}$ will consist of certain subsets of W. To specify these subsets we define $W' = S_W \times B$, where S_W contains all unary linear polynomials (aka sections) over $(W, \circ, \bigcirc\!\!\!\!\wedge, 1)$. Also we define and

$$x \sqsubseteq (u, b) \text{ iff } u(x) \leq_{\mathbf{A}} b.$$

Then $\mathbf{W} = (W, W', \sqsubseteq)$ is an example of a *lattice frame*. (Dedekind, McNeille, Birkhoff) These play the role of *Kripke frames* for non-distributive logics. We have two set of worlds: W for the join-irreducibles and W' for the meet-irreducibles.

The *Galois algebra* of $\mathbf{W}$ is $\mathbf{W}^+ = (\mathcal{P}(W)_{\gamma_\sqsubseteq}, \cap, \cup_{\gamma_\sqsubseteq})$ and it is a complete lattice. For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^\rhd = \{b \in W' : x \sqsubseteq b, \text{ for all } x \in X\}$$
$$Y^\lhd = \{a \in W : a \sqsubseteq y, \text{ for all } y \in Y\}$$

$\gamma_\sqsubseteq : \mathcal{P}(W) \to \mathcal{P}(W)$, $\gamma_\sqsubseteq(X) = X^{\rhd\lhd}$, is a closure operator.

Residuated frames

In our case, we have more structure and $\mathbf{W}^+$ becomes a residuated lattice.

Residuated frames

In our case, we have more structure and $\mathbf{W}^+$ becomes a residuated lattice.

$(W, \circ, 1)$ is a monoid.

Residuated frames

In our case, we have more structure and $\mathbf{W}^+$ becomes a residuated lattice.

$(W, \circ, 1)$ is a monoid. Also, W acts (as a monoid) on $W' = S_W \times B \equiv W \times B \times W$ on both sides.

In our case, we have more structure and $\mathbf{W}^+$ becomes a residuated lattice.

$(W, \circ, 1)$ is a monoid. Also, W acts (as a monoid) on $W' = S_W \times B \equiv W \times B \times W$ on both sides. Also, the relation $\sqsubseteq$ connects the monoid operation with the actions: it satisfies the *nuclear* condition:

$$(x \circ y) \sqsubseteq z \;\Leftrightarrow\; y \sqsubseteq (x \mathbin{\backslash\!\!\backslash} z) \;\Leftrightarrow\; x \sqsubseteq (z \mathbin{/\!\!/} y)$$

Here $x \mathbin{\backslash\!\!\backslash} (u, b) = (u \circ x, b)$.

Residuated frames

In our case, we have more structure and $\mathbf{W}^+$ becomes a residuated lattice.

$(W, \circ, 1)$ is a monoid. Also, W acts (as a monoid) on $W' = S_W \times B \equiv W \times B \times W$ on both sides. Also, the relation $\sqsubseteq$ connects the monoid operation with the actions: it satisfies the *nuclear* condition:

$$(x \circ y) \sqsubseteq z \;\Leftrightarrow\; y \sqsubseteq (x \mathbin{\backslash\!\!\backslash} z) \;\Leftrightarrow\; x \sqsubseteq (z \mathbin{/\!\!/} y)$$

Here $x \mathbin{\backslash\!\!\backslash} (u, b) = (u \circ x, b)$.

Then $\mathbf{W}^+$ is a residuated lattice (NG - P. Jipsen), where multiplication is given by: $X \circ_\gamma Y = \gamma(X \circ Y)$.

(This is because γ_N is a nucleus.)

Distributive frames

We also have additional structure, as W acts on W' with actions corresponding to $\textcircled{\wedge}$, as well.

We also have additional structure, as W acts on W' with actions corresponding to $\owedge$, as well.

$$(x \owedge y)\ N\ w \;\Leftrightarrow\; y\ N\ (x \mathbin{\backslash\!\!\!\!\bigcirc} w) \;\Leftrightarrow\; x\ N\ (w \oslash y)$$

$$\frac{x \owedge (y \owedge w) \sqsubseteq z}{(x \owedge y) \owedge w \sqsubseteq z}\ (\owedge a) \qquad \frac{x \owedge y \sqsubseteq z}{y \owedge x \sqsubseteq z}\ (\owedge e)$$

$$\frac{x \sqsubseteq z}{x \owedge y \sqsubseteq z}\ (\owedge i) \qquad \frac{x \owedge x \sqsubseteq z}{x \sqsubseteq z}\ (\owedge c)$$

We also have additional structure, as W acts on W' with actions corresponding to $\owedge$, as well.

$$(x \owedge y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \mathbin{\backslash\!\!\!\bigcirc} w) \ \Leftrightarrow\ x\ N\ (w \mathbin{/\!\!\!\bigcirc} y)$$

$$\frac{x \owedge (y \owedge w) \sqsubseteq z}{(x \owedge y) \owedge w \sqsubseteq z}\ (\owedge a) \qquad \frac{x \owedge y \sqsubseteq z}{y \owedge x \sqsubseteq z}\ (\owedge e)$$

$$\frac{x \sqsubseteq z}{x \owedge y \sqsubseteq z}\ (\owedge i) \qquad \frac{x \owedge x \sqsubseteq z}{x \sqsubseteq z}\ (\owedge c)$$

Results in [G - Jipsen] guarantee that $\mathbf{W}^+$ is a distributive residuated lattice. (This is because γ_N is a distributive nucleus; in particular, $\owedge_{\gamma_\sqsubseteq} = \cap$.)

The embedding

In our case, we have further structure: B is a partial algebra and copies of B sit inside both W and W' ($b \equiv (id, b)$). Furthermore, $\sqsubseteq$ satisfies special properties reminiscent of a proof-theoretic sequent calculus for distributive $\mathbf{FL}$.

The embedding

In our case, we have further structure: B is a partial algebra and copies of B sit inside both W and W' ($b \equiv (id, b)$). Furthermore, $\sqsubseteq$ satisfies special properties reminiscent of a proof-theoretic sequent calculus for distributive $\mathbf{FL}$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

The embedding

In our case, we have further structure: B is a partial algebra and copies of B sit inside both W and W' ($b \equiv (id, b)$). Furthermore, $\sqsubseteq$ satisfies special properties reminiscent of a proof-theoretic sequent calculus for distributive $\mathbf{FL}$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

Theorem. [G.-Jipsen] Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^{+}$, $b \mapsto \{b\}^{\triangleleft} = \{b\}^{\triangleright\triangleleft}$ is a homomorphism.
I.e., $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{W}^{+}} \{b\}^{\triangleleft}$, for all $a, b \in B$. ($\bullet$ is a connective)

In our case, we have further structure: B is a partial algebra and copies of B sit inside both W and W' ($b \equiv (id, b)$). Furthermore, $\sqsubseteq$ satisfies special properties reminiscent of a proof-theoretic sequent calculus for distributive $\mathbf{FL}$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

Theorem. [G.-Jipsen] Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^{+}, \ b \mapsto \{b\}^{\triangleleft} = \{b\}^{\triangleright\triangleleft}$ is a homomorphism.
I.e., $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{W}^{+}} \{b\}^{\triangleleft}$, for all $a, b \in B$. ($\bullet$ is a connective)

If $\sqsubseteq$ is antysymmetric, then the map is an embedding.

In our case, we have further structure: B is a partial algebra and copies of B sit inside both W and W' ($b \equiv (id, b)$). Furthermore, $\sqsubseteq$ satisfies special properties reminiscent of a proof-theoretic sequent calculus for distributive $\mathbf{FL}$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

Theorem. [G.-Jipsen] Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $\{\}^{\lhd} : \mathbf{B} \to \mathbf{W}^+, \;\; b \mapsto \{b\}^{\lhd} = \{b\}^{\rhd\lhd}$ is a homomorphism.
I.e., $\{a \bullet_{\mathbf{B}} b\}^{\lhd} = \{a\}^{\lhd} \bullet_{\mathbf{W}^+} \{b\}^{\lhd}$, for all $a, b \in B$. ($\bullet$ is a connective)

If $\sqsubseteq$ is antysymmetric, then the map is an embedding.

In the following slide, $a, b \in B$; $x, y \in W$; $z \in W'$.

DGN

$$\frac{x \sqsubseteq a \quad a \sqsubseteq z}{x \sqsubseteq z}\ (\mathsf{CUT}) \qquad \frac{}{a \sqsubseteq a}\ (\mathsf{Id})$$

$$\frac{x \owedge (y \owedge w) \sqsubseteq z}{(x \owedge y) \owedge w \sqsubseteq z}\ (\owedge a) \qquad \frac{x \owedge y \sqsubseteq z}{y \owedge x \sqsubseteq z}\ (\owedge e)$$

$$\frac{x \sqsubseteq z}{x \owedge y \sqsubseteq z}\ (\owedge i) \qquad \frac{x \owedge x \sqsubseteq z}{x \sqsubseteq z}\ (\owedge c)$$

$$\frac{x \sqsubseteq a \quad b \sqsubseteq z}{x \circ (a \backslash b) \sqsubseteq z}\ (\backslash \mathsf{L}) \qquad \frac{a \circ x \sqsubseteq b}{x \sqsubseteq a \backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x \sqsubseteq a \quad b \sqsubseteq z}{(b/a) \circ x \sqsubseteq z}\ (/\mathsf{L}) \qquad \frac{x \circ a \sqsubseteq b}{x \sqsubseteq b/a}\ (/\mathsf{R})$$

$$\frac{a \circ b \sqsubseteq z}{a \cdot b \sqsubseteq z}\ (\cdot\mathsf{L}) \qquad \frac{x \sqsubseteq a \quad y \sqsubseteq b}{x \circ y \sqsubseteq a \cdot b}\ (\cdot\mathsf{R}) \qquad \frac{\varepsilon \sqsubseteq z}{1 \sqsubseteq z}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \sqsubseteq 1}\ (1\mathsf{R})$$

$$\frac{a \owedge b \sqsubseteq z}{a \wedge b \sqsubseteq z}\ (\wedge \mathsf{L}\ell) \qquad \frac{x \sqsubseteq a \quad x \sqsubseteq b}{x \sqsubseteq a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{a \sqsubseteq z \quad b \sqsubseteq z}{a \vee b \sqsubseteq z}\ (\vee \mathsf{L}) \qquad \frac{x \sqsubseteq a}{x \sqsubseteq a \vee b}\ (\vee \mathsf{R}\ell) \qquad \frac{x \sqsubseteq b}{x \sqsubseteq a \vee b}\ (\vee \mathsf{R}r)$$

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1 x_2 \wedge y) \vee (x_2 x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1 x_2 \wedge y) \vee (x_2 x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

$x_1 x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1x_2 \wedge y) \vee (x_2x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$x_1x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$$\frac{x_1 \wedge y \leq z \quad x_2 \wedge y \leq z \quad yx_1 \leq z \quad yx_2 \leq z}{x_1x_2 \wedge y \leq z}$$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1x_2 \wedge y) \vee (x_2x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$x_1x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$$\frac{x_1 \wedge y \leq z \quad x_2 \wedge y \leq z \quad yx_1 \leq z \quad yx_2 \leq z}{x_1x_2 \wedge y \leq z}$$

$$\frac{x_1 \owedge y \sqsubseteq z \quad x_2 \owedge y \sqsubseteq z \quad y \circ x_1 \sqsubseteq z \quad y \circ x_2 \sqsubseteq z}{x_1 \circ x_2 \owedge y \sqsubseteq z} \ R(\varepsilon)$$

Structural rules

Given a linearized equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms and t_0 is linear, we construct the rule $R(\varepsilon)$

$$\frac{t_1 \sqsubseteq z \quad \cdots \quad t_n \sqsubseteq z}{t_0 \sqsubseteq z}\ (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$ and z in W'.

Structural rules

Given a linearized equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms and t_0 is linear, we construct the rule $R(\varepsilon)$

$$\frac{t_1 \sqsubseteq z \quad \cdots \quad t_n \sqsubseteq z}{t_0 \sqsubseteq z} \; (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$ and z in W'.

Theorem. [G.-Jipsen] If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\wedge, \vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Free algebra

Let $(F, \circ, \varepsilon, \owedge)$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$.

Free algebra

Let $(F, \circ, \varepsilon, \bigotimes)$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators or ε; also we stipulate $x \leq^F \varepsilon$.

Free algebra

Let $(F, \circ, \varepsilon, \bigcirc\!\!\!\!\wedge)$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators or ε; also we stipulate $x \leq^F \varepsilon$. We denote by $\mathbf{F}$ the resulting ordered algebra.

Free algebra

Let $(F, \circ, \varepsilon, \textcircled{\wedge})$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators or ε; also we stipulate $x \leq^F \varepsilon$. We denote by $\mathbf{F}$ the resulting ordered algebra.

By Higman's Lemma, F is dually well-ordered (it has no infinite antichains and no infinite ascending chains).

Free algebra

Let $(F, \circ, \varepsilon, \bigcircledwedge)$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators or ε; also we stipulate $x \leq^F \varepsilon$. We denote by $\mathbf{F}$ the resulting ordered algebra.

By Higman's Lemma, F is dually well-ordered (it has no infinite antichains and no infinite ascending chains).

$\mathbf{F}$ is residuated in a stong sense:

Lemma For all $x \in F$, $u \in S_F$ and $b \in B$, $u(x) \leq^F b$ iff $x \leq^F \frac{b}{u}$.

Free algebra

Let $(F, \circ, \varepsilon, \owedge)$ be the free algebra over $|B|$-many generators, where ε is a unit for $\circ$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators or ε; also we stipulate $x \leq^F \varepsilon$. We denote by $\mathbf{F}$ the resulting ordered algebra.

By Higman's Lemma, F is dually well-ordered (it has no infinite antichains and no infinite ascending chains).

$\mathbf{F}$ is residuated in a stong sense:

Lemma For all $x \in F$, $u \in S_F$ and $b \in B$, $u(x) \leq^F b$ iff $x \leq^F \frac{b}{u}$.

where $\frac{z}{u}$ is defined by induction on the structure of u by:

$$\frac{z}{id} = z,\ \frac{z}{u \circ y} = \frac{z /\!\!/ y}{u},\ \frac{z}{y \circ u} = \frac{y \backslash\!\!\backslash z}{u},\ \frac{z}{u \owedge y} = \frac{z \oslash y}{u} \text{ and } \frac{z}{y \owedge u} = \frac{y \obslash z}{u},$$

where id is the identity section and where $\backslash\!\!\backslash, /\!\!/$ are the residuals of $\circ$ and $\obslash, \oslash$ are the residuls of $\owedge$ in $\mathbf{F}$.

Finiteness

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite.

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Consider $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ \sqsubseteq, \cdot^{\mathbf{F}}, \backslash\!\!\backslash_h, /\!\!/_h, \{1\})$, where $x\ (h \circ \sqsubseteq)\ z$ iff $h(x) \sqsubseteq z$, $x \backslash\!\!\backslash_h z = h(x) \backslash\!\!\backslash z$ and $z /\!\!/_h y = z /\!\!/ h(y)$.

Finiteness

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Consider $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ \sqsubseteq, \cdot^{\mathbf{F}}, \backslash\!\!\backslash_h, /\!\!/_h, \{1\})$, where $x\ (h \circ \sqsubseteq)\ z$ iff $h(x) \sqsubseteq z$, $x \backslash\!\!\backslash_h z = h(x) \backslash\!\!\backslash z$ and $z /\!\!/_h y = z /\!\!/ h(y)$.

Claim 1: $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a distributive residuated frame.

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Consider $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ \sqsubseteq, \cdot^{\mathbf{F}}, \mathbin{\backslash\!\!\backslash}_h, /\!\!/_h, \{1\})$, where $x\ (h \circ \sqsubseteq)\ z$ iff $h(x) \sqsubseteq z$, $x \mathbin{\backslash\!\!\backslash}_h z = h(x) \mathbin{\backslash\!\!\backslash} z$ and $z /\!\!/_h y = z /\!\!/ h(y)$.

Claim 1: $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a distributive residuated frame.

To prove that $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $\{z\}^{\lhd_\sqsubseteq} = \{x \in W : x \sqsubseteq z\}$, for $z \in W'$.

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Consider $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ \sqsubseteq, \cdot^{\mathbf{F}}, \backslash\!\!\backslash_h, /\!\!/_h, \{1\})$, where $x\ (h \circ \sqsubseteq)\ z$ iff $h(x) \sqsubseteq z$, $x \backslash\!\!\backslash_h z = h(x) \backslash\!\!\backslash z$ and $z /\!\!/_h y = z /\!\!/ h(y)$.

Claim 1: $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a distributive residuated frame.

To prove that $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $\{z\}^{\lhd_\sqsubseteq} = \{x \in W : x \sqsubseteq z\}$, for $z \in W'$.

Claim 2: $h[\{z\}^{\lhd}] = \{z\}^{\lhd_\sqsubseteq}$

Indeed, for all $x \in W$, there is $x' \in F$ with $h(x') = x$, as h is surjective; so, $x = h(x') \in \{(u,b)\}^{\lhd_N}$ iff $x' \in \{(u,b)\}^{\lhd}$, hence $x \in h[\{(u,b)\}^{\lhd}]$. Conversely, if $x \in h[\{(u,b)\}^{\lhd}]$, then $x = h(x')$ for some $x' \in \{(u,b)\}^{\lhd}$, hence $x = h(x') \in \{(u,b)\}^{\lhd_\sqsubseteq}$.

Theorem If $\mathbf{A}$ is an IDRL and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, then $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite.

Proof (Sketch) Note that the (surjective) homomorphism $h : F \to W$ that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B is order-preserving.

Consider $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ \sqsubseteq, \cdot^{\mathbf{F}}, \backslash\!\!\backslash_h, /\!\!/_h, \{1\})$, where $x\ (h \circ \sqsubseteq)\ z$ iff $h(x) \sqsubseteq z$, $x \backslash\!\!\backslash_h z = h(x) \backslash\!\!\backslash z$ and $z /\!\!/_h y = z /\!\!/ h(y)$.

Claim 1: $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a distributive residuated frame.

To prove that $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $\{z\}^{\lhd_\sqsubseteq} = \{x \in W : x \sqsubseteq z\}$, for $z \in W'$.

Claim 2: $h[\{z\}^{\lhd}] = \{z\}^{\lhd_\sqsubseteq}$
Indeed, for all $x \in W$, there is $x' \in F$ with $h(x') = x$, as h is surjective; so, $x = h(x') \in \{(u,b)\}^{\lhd_N}$ iff $x' \in \{(u,b)\}^{\lhd}$, hence $x \in h[\{(u,b)\}^{\lhd}]$. Conversely, if $x \in h[\{(u,b)\}^{\lhd}]$, then $x = h(x')$ for some $x' \in \{(u,b)\}^{\lhd}$, hence $x = h(x') \in \{(u,b)\}^{\lhd_\sqsubseteq}$.

So, it suffices to show that there are finitely many sets of the from $\{z\}^{\lhd} = \{x \in F : x\ (h \circ \sqsubseteq)\ z\}$, for $z \in W'$.

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow\{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.

Proof (cont)

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow \{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.Equivalently, $v(x) \in h^{-1}(\downarrow_A b)$, for some $v \in h^{-1}(u)$.

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow \{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.Equivalently, $v(x) \in h^{-1}(\downarrow_A b)$, for some $v \in h^{-1}(u)$.Since h is order-reserving, $h^{-1}(\downarrow_A b)$ is a downset in $\mathbf{F}$ and, because $\mathbf{F}$ is dually well-ordered, this downset is equal to $\downarrow M_b$, for some *finite* $M_b \subseteq F$.

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow \{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.Equivalently, $v(x) \in h^{-1}(\downarrow_A b)$, for some $v \in h^{-1}(u)$.Since h is order-reserving, $h^{-1}(\downarrow_A b)$ is a downset in $\mathbf{F}$ and, because $\mathbf{F}$ is dually well-ordered, this downset is equal to $\downarrow M_b$, for some *finite* $M_b \subseteq F$. So, the above statement is equivalent to $v(x) \leq m$, or to $x \leq \frac{m}{v}$, for some $v \in h^{-1}(u)$ and some $m \in M_b$.

Proof (cont)

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow \{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.Equivalently, $v(x) \in h^{-1}(\downarrow_A b)$, for some $v \in h^{-1}(u)$.Since h is order-reserving, $h^{-1}(\downarrow_A b)$ is a downset in $\mathbf{F}$ and, because $\mathbf{F}$ is dually well-ordered, this downset is equal to $\downarrow M_b$, for some *finite* $M_b \subseteq F$. So, the above statement is equivalent to $v(x) \leq m$, or to $x \leq \frac{m}{v}$, for some $v \in h^{-1}(u)$ and some $m \in M_b$.

Claim 4: $\{\frac{m}{v} : m \in M_b, b \in B, h(v) = u, u \in S_W\}$ is finite.
Indeed, it is a subset of the finite set $\uparrow \bigcup_{b \in B} M_b$, as $m \leq \frac{m}{v}$ (or $v(m) \leq m$), by integrality. Thus, there are only finitely many choices for $\{(u,b)\}^{\lhd}$.

Proof (cont)

Claim 3: $\{(u,b)\}^{\lhd} = \downarrow \{\frac{m}{v} : m \in M_b, h(v) = u\}$, where M_b is a finite subset of F.
Indeed, for $x \in F$, and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism.Equivalently, $v(x) \in h^{-1}(\downarrow_A b)$, for some $v \in h^{-1}(u)$.Since h is order-reserving, $h^{-1}(\downarrow_A b)$ is a downset in $\mathbf{F}$ and, because $\mathbf{F}$ is dually well-ordered, this downset is equal to $\downarrow M_b$, for some *finite* $M_b \subseteq F$. So, the above statement is equivalent to $v(x) \leq m$, or to $x \leq \frac{m}{v}$, for some $v \in h^{-1}(u)$ and some $m \in M_b$.

Claim 4: $\{\frac{m}{v} : m \in M_b, b \in B, h(v) = u, u \in S_W\}$ is finite.
Indeed, it is a subset of the finite set $\uparrow \bigcup_{b \in B} M_b$, as $m \leq \frac{m}{v}$ (or $v(m) \leq m$), by integrality. Thus, there are only finitely many choices for $\{(u,b)\}^{\lhd}$.

Corollary Every variety of integral distributive residuated lattices axiomatized by equations over the signature $\{\wedge, \vee, \cdot, 1\}$ has the FEP.