CSP for Commutative, Idempotent Groupoids

David Failing and Clifford Bergman

Department of Mathematics
Iowa State University
Ames, IA 50011

AMS Spring Western Section Meeting
April 14, 2013

Constraint Satisfaction Problem

Definition

An instance of the CSP is a triple $\mathcal{R} = (V, \mathbf{A}, \mathcal{C})$ in which:

- V is a finite set of variables,
- $\mathbf{A}$ is a finite, idempotent algebra
- $\mathcal{C} = \{(S_i, R_i) \mid i = 1, \ldots, n\}$ is a set of constraints, with $S_i \subseteq V$ and $R_i \leq \mathbf{A}^{S_i}$.

A solution to $\mathcal{R}$ is a map $f \colon V \to A$ such that for all i, $f(S_i) \in R_i$. The algebra $\mathbf{A}$ is said to be tractable if the decision problem CSP($\mathbf{A}$) is in P. A variety $\mathcal{V}$ is tractable if every finite algebra in $\mathcal{V}$ is tractable.

Known Results

Theorem (Bulatov and Dalmau)

The variety of quasigroups is tractable.

Definition

An algebra is congruence meet-semidistributive (SD($\wedge$)) if its congruence lattice satisfies

$$(x \wedge y \approx x \wedge z) \Rightarrow (x \wedge (y \vee z) \approx x \wedge y)$$

Theorem (Barto and Kozik)

An SD($\wedge$) *variety is tractable.*

Theorem (Jeavons, Cohen, Gyssens '97)

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The CSP Dichotomy...

Theorem (Bulatov, Jeavons, Krokhin '05; Maroti & McKenzie '08)

Let **A** *be a finite idempotent algebra. If* **A** *has no weak near-unanimity term (WNU), then* **A** *is NP-complete.*

Algebraic Dichotomy Conjecture

If **A** has a WNU term, then it is tractable.

Motivation:

- A binary operation is a WNU if and only if is commutative and idempotent.
- Adding associativity suffices for tractability of an algebra.
- Any weakening of associativity should also suffice.

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Definition

Let $\mathbf{A} = \langle A, \cdot \rangle$ be a groupoid. We call $\mathbf{A}$ a CI-groupoid if $\cdot$ is both commutative and idempotent. Usually, we write xy for $x \cdot y$.

The Moufang Law $x(y(zy)) = ((xy)z)y$ is one weakening of associativity.

Definition

An identity $p \approx q$ is of Bol-Moufang type if (i) the only operation in p, q is $\cdot$, (ii) the same three variables appear on both sides, in the same order, (iii) one of the variables appears twice (iv) the remaining two variables appear only once.

- There are 60 such identities. Which ones are equivalent with respect to C+I?

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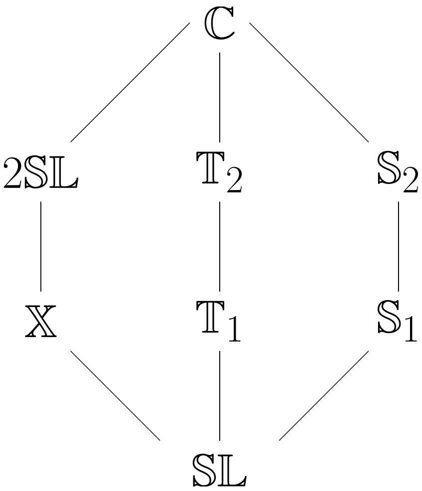
The 8 Varieties of CI-Groupoids of Bol-Moufang Type



The Variety $\mathcal{S}_2$ of Bol-Moufang CI-Groupoids

Definition

$\mathcal{S}_2$ is the variety of CI-groupoids satisfying $x(y(xz)) \approx x((yx)z)$.

Theorem (KKVW '13)

A finite idempotent algebra with WNU terms $v(x, y, z)$ and $w(x, y, z, u)$ such that $v(y, x, x) \approx w(y, x, x, x)$ is SD($\wedge$).

Theorem

$\mathcal{S}_2$ is tractable.

Proof.

$\mathcal{S}_2$ has WNU terms $v(x, y, z) = (xy)(z(xy))$ and $w(x, y, z, u) = (xy)(zu)$ such that $v(y, x, x) \approx w(y, x, x, x)$. □

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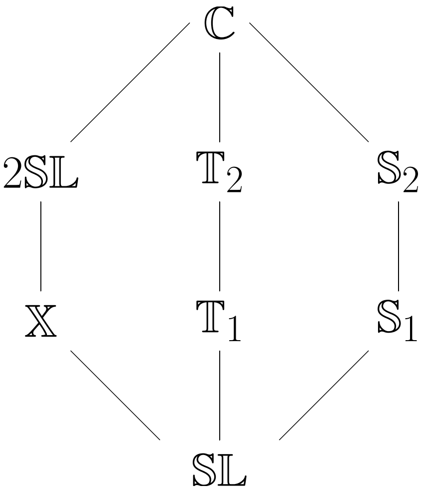

The Płonka Sum of Groupoids

Definition

Given

- $\mathbf{S} = \langle S, \vee \rangle$ a semilattice,
- $\{\mathbf{A}_s \mid s \in S\}$ a set of groupoids, and
- $\{\phi_{s,t} : \mathbf{A}_s \to \mathbf{A}_t \mid s \leq_\vee t\}$ a set of "nice" homomorphisms,

the **Płonka sum** over S of the groupoids $\{\mathbf{A}_s : s \in S\}$ is the groupoid $\mathbf{A}$ with universe $\biguplus_{s \in S} A_s$ and multiplication given by:

$$x_1 *^{\mathbf{A}} x_2 = \phi_{s_1,s}(x_1) *^{\mathbf{A}_s} \phi_{s_2,s}(x_2)$$

where $x_i \in \mathbf{A}_{s_i}$, $s = s_1 \vee s_2$.

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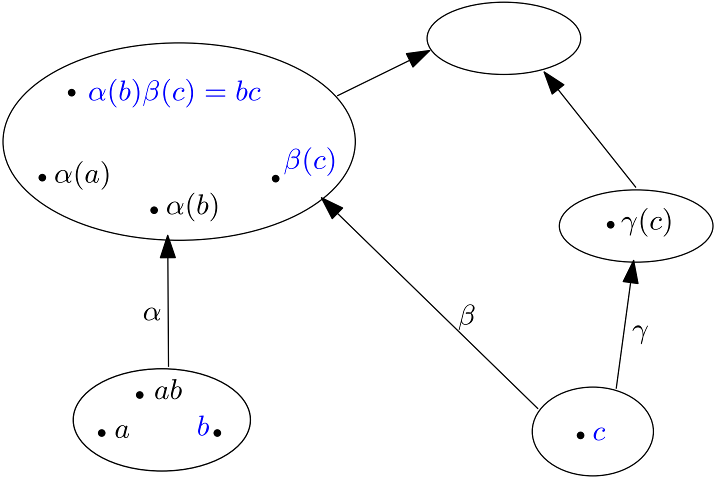

Płonka's Theorem

Theorem

Let $\mathcal{V}$ be the variety of groupoids defined by $\Sigma \cup \{x \vee y \approx x\}$ for some term $x \vee y$ and set Σ of regular identities. The following classes of algebras coincide:

(1) *The class $\mathbf{Pl}(\mathcal{V})$of Płonka sums of $\mathcal{V}$-algebras.*

(2) *The variety of algebras of type ρ defined by the identities Σ and the following identities:*

$$x \vee x \approx x \tag{P1}$$

$$(x \vee y) \vee z \approx x \vee (y \vee z) \tag{P2}$$

$$x \vee (y \vee z) \approx x \vee (z \vee y) \tag{P3}$$

$$x \vee (y * z) \approx x \vee y \vee z \tag{P4}$$

$$(x * y) \vee z \approx (x \vee z) * (y \vee z) \tag{P5}$$

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Pseudopartition Operations

Definition

A term $x \vee y$ satisfying $(P1)$-$(P4)$ is a pseudopartition operation. The congruence on an algebra possessing such a term defined by

$$a \, \sigma \, b \Leftrightarrow [a \vee b = a \text{ and } b \vee a = b]$$

is known as the semilattice replica congruence.

Theorem (Main Result)

Let $\mathbf{A}$ *be a finite idempotent algebra with pseudopartition operation* $x \vee y$*, such that every block of its semilattice replica congruence lies in the same tractable variety. Then* $\mathbf{A}$ *is tractable.*

Squags and $\mathcal{T}_2$

Definition

$\mathcal{T}_2$ is the variety of CI-groupoids satisfying $x(y(yz)) \approx ((xy)y)z$.

Definition

The variety of Steiner quasigroups (squags) is the variety of CI-groupoids satisfying $y(xy) \approx x$.

Theorem

$\mathcal{T}_2$ is tractable.

Proof.

Let $x \vee y \approx y(xy)$ in $\mathcal{T}_2$. Each σ-class is a squag. □

Theorem

The subvariety $\mathcal{T}_1$ (defined by $x(x(yz)) \approx (x(xy))z$) of $\mathcal{T}_2$ is the class of Płonka sum of squags.

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CID and CIE Groupoids

Definition

A groupoid is distributive (D) if it satisfies $x(yz) \approx (xy)(xz)$. It is entropic (E) if it satisfies $(xy)(zw) \approx (xz)(yw)$.

Theorem

Every finite CID-groupoid (and hence CIE-groupoid) is a Płonka sum of quasigroups.

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The variety of CID-groupoids is tractable.

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