DEDEKIND'S TRANSPOSITION PRINCIPLE

AND

ISOTOPIC ALGEBRAS WITH NONISOMORPHIC CONGRUENCE LATTICES

William DeMeo

williamdemeo@gmail.com

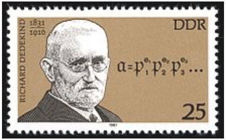
University of South Carolina

AMS Spring Western Sectional Meeting
University of Colorado, Boulder, CO

April 13-14, 2013

These slides and other resources are available at

http://williamdemeo.wordpress.com



Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.

Dedekind's Transposition Principle

for modular lattices

Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.

Theorem (Dedekind's Transposition Principle)

$\mathbf{L}$ is modular iff for all $a, b \in L$ the maps φ_a and ψ_b are inverse lattice isomorphisms of $[\![a \wedge b, a]\!]$ and $[\![b, a \vee b]\!]$.

Dedekind's Transposition Principle

for modular lattices

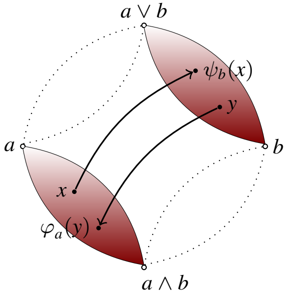
Notation

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice with $a \in L$.

Let φ_a and ψ_a be the *perspectivity maps*

$$\varphi_a(x) = x \wedge a \quad \text{and} \quad \psi_a(x) = x \vee a$$

For $x, y \in L$, let $[\![x, y]\!]_L = \{z \in L \mid x \leqslant z \leqslant y\}$.



Theorem (Dedekind's Transposition Principle)

$\mathbf{L}$ *is modular iff for all* $a, b \in L$ *the maps* φ_a *and* ψ_b *are inverse lattice isomorphisms of* $[\![a \wedge b, a]\!]$ *and* $[\![b, a \vee b]\!]$.

Another transposition principle

for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

If L is a sublattice of $\operatorname{Eq} X$ with $\eta, \theta \in L$, then we define

$$[\![\eta, \theta]\!]_L = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta\}.$$

Another transposition principle

for lattices of equivalence relations

Let X be a set and let $\mathrm{Eq}\,X$ be the lattice of equivalence relations on X.

If L is a sublattice of $\mathrm{Eq}\,X$ with $\eta, \theta \in L$, then we define

$$[\![\eta, \theta]\!]_L = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta\}.$$

For $\beta \in \mathrm{Eq}\,X$, let $[\![\eta, \theta]\!]_L^\beta$ be the set of relations in $[\![\eta, \theta]\!]_L$ that permute with β,

$$[\![\eta, \theta]\!]_L^\beta = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta \text{ and } \gamma \circ \beta = \beta \circ \gamma\}.$$

Another transposition principle

for lattices of equivalence relations

Let X be a set and let $\operatorname{Eq} X$ be the lattice of equivalence relations on X.

If L is a sublattice of $\operatorname{Eq} X$ with $\eta, \theta \in L$, then we define

$$[\![\eta, \theta]\!]_L = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta\}.$$

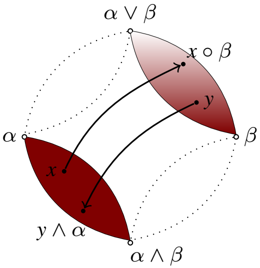
For $\beta \in \operatorname{Eq} X$, let $[\![\eta, \theta]\!]_L^\beta$ be the set of relations in $[\![\eta, \theta]\!]_L$ that permute with β,

$$[\![\eta, \theta]\!]_L^\beta = \{\gamma \in L \mid \eta \leqslant \gamma \leqslant \theta \text{ and } \gamma \circ \beta = \beta \circ \gamma\}.$$

Lemma

Suppose α and β are permuting relations in $L \leqslant \operatorname{Eq} X$.

Then $[\![\beta, \alpha \vee \beta]\!]_L \cong [\![\alpha \wedge \beta, \alpha]\!]_L^\beta \leqslant [\![\alpha \wedge \beta, \alpha]\!]_L$.



Dedekind's Rule

The proof requires the following version of *Dedekind's Rule:*

Lemma

Suppose $\alpha, \beta, \gamma \in L \leqslant \mathrm{Eq}\, X$ *and* $\alpha \leqslant \beta$*.*

Then the following identities of subsets of X^2 *hold:*

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma)$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha)$$

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e.} \quad (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e.} \quad (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some $\mathbf{C}$.

It is easy to verify that $\sim$ is an equivalence relation.

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e.} \quad (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some $\mathbf{C}$.

If $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ and $\mathrm{Con}(\mathbf{A} \times \mathbf{C})$ happens to be modular, then we write $\mathbf{A} \sim_{\mathbf{C}}^{\mathrm{mod}} \mathbf{B}$ and say that $\mathbf{A}$ and $\mathbf{B}$ are *modular isotopic over* $\mathbf{C}$.

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e. } (\forall a \in A)\,(\forall c \in C) \quad \varphi(a,c) = (\varphi_1(a,c), c)$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some $\mathbf{C}$.

If $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ and $\mathrm{Con}(\mathbf{A} \times \mathbf{C})$ happens to be modular, then we write $\mathbf{A} \sim_{\mathbf{C}}^{\mathrm{mod}} \mathbf{B}$ and say that $\mathbf{A}$ and $\mathbf{B}$ are *modular isotopic over* $\mathbf{C}$.

We call $\mathbf{A}$ and $\mathbf{B}$ *modular isotopic in one step*, denoted $\mathbf{A} \sim_{1}^{\mathrm{mod}} \mathbf{B}$, if they are modular isotopic over some $\mathbf{C}$.

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be algebras of the same type.

$\mathbf{A}$ and $\mathbf{B}$ are *isotopic over* $\mathbf{C}$, denoted $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$, if there is an isomorphism

$$\varphi : \mathbf{A} \times \mathbf{C} \xrightarrow{\cong} \mathbf{B} \times \mathbf{C} \quad \text{that leaves the second coordinate fixed}$$

$$\text{i.e.} \quad (\forall a \in A)\,(\forall c \in C) \quad \varphi(a, c) = (\varphi_1(a, c), c)$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *isotopic*, denoted $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ for some $\mathbf{C}$.

If $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$ and $\operatorname{Con}(\mathbf{A} \times \mathbf{C})$ happens to be modular, then we write $\mathbf{A} \sim_{\mathbf{C}}^{\text{mod}} \mathbf{B}$ and say that $\mathbf{A}$ and $\mathbf{B}$ are *modular isotopic over* $\mathbf{C}$.

We call $\mathbf{A}$ and $\mathbf{B}$ *modular isotopic in one step*, denoted $\mathbf{A} \sim_{1}^{\text{mod}} \mathbf{B}$, if they are modular isotopic over some $\mathbf{C}$.

We call $\mathbf{A}$ and $\mathbf{B}$ are *modular isotopic*, denoted $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$, if $(\mathbf{A}, \mathbf{B})$ is in the transitive closure of $\sim_{1}^{\text{mod}}$.

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con}\mathbf{A} \cong \operatorname{Con}\mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$?

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$?

As you have guessed, the answer is no!

The perspectivity map that is so useful when $\operatorname{Con}(\mathbf{A} \times \mathbf{C})$ is modular can fail *miserably* in the non-modular case...

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con} \mathbf{A} \cong \operatorname{Con} \mathbf{B}$?

As you have guessed, the answer is no!

The perspectivity map that is so useful when $\operatorname{Con}(\mathbf{A} \times \mathbf{C})$ is modular can fail *miserably* in the non-modular case... *even when* $\mathbf{A} \cong \mathbf{B}$!

Lemma 11. If $\mathbf{A} \sim^{\text{mod}} \mathbf{B}$ then $\operatorname{Con}\mathbf{A} \cong \operatorname{Con}\mathbf{B}$.

The proof is a nice/easy application of Dedekind's Transposition Principle.

Could we use the same strategy with the non-modular version of the transposition principle to show that $\mathbf{A} \sim \mathbf{B}$ implies $\operatorname{Con}\mathbf{A} \cong \operatorname{Con}\mathbf{B}$?

As you have guessed, the answer is no!

The perspectivity map that is so useful when $\operatorname{Con}(\mathbf{A} \times \mathbf{C})$ is modular can fail *miserably* in the non-modular case... *even when* $\mathbf{A} \cong \mathbf{B}$!

But this only shows that the same argument doesn't work...

COUNTEREXAMPLES

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con}\mathbf{A} \ncong \operatorname{Con}\mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

Counterexamples

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con}\mathbf{A} \ncong \operatorname{Con}\mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

For any group G, let $\operatorname{Sub}(G)$ denote the lattice of subgroups of G.

Counterexamples

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con} \mathbf{A} \ncong \operatorname{Con} \mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

For any group G, let $\operatorname{Sub}(G)$ denote the lattice of subgroups of G.

A group G is called a *Dedekind group* if every subgroup of G is normal.

Counterexamples

We describe a class of examples in which $\mathbf{A} \sim \mathbf{B}$ and $\operatorname{Con}\mathbf{A} \ncong \operatorname{Con}\mathbf{B}$.

The examples show that congruence lattices of isotopic algebras can differ arbitrarily in size.

For any group G, let $\operatorname{Sub}(G)$ denote the lattice of subgroups of G.

A group G is called a *Dedekind group* if every subgroup of G is normal.

Let S be any group and let D denote the *diagonal subgroup* of $S \times S$,

$$D = \{(x,x) \mid x \in S\}$$

The interval $[\![D, S \times S]\!] \leqslant \operatorname{Sub}(S \times S)$ is described by the following

Lemma

The filter above the diagonal subgroup of $S \times S$ is isomorphic to the lattice of normal subgroups of S.

The example

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let $D = \{(x_1, x_2) \in G \mid x_1 = x_2\}, \quad T_1 = S_1 \times \langle 1 \rangle, \quad T_2 = \langle 1 \rangle \times S_2$.

The example

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let $D = \{(x_1, x_2) \in G \mid x_1 = x_2\}, \quad T_1 = S_1 \times \langle 1 \rangle, \quad T_2 = \langle 1 \rangle \times S_2$.

Then $D \cong T_1 \cong T_2$, and these are pair-wise compliments:

$$\langle T_1, T_2 \rangle = \langle T_1, D \rangle = \langle D, T_2 \rangle = G$$

$$T_1 \cap D = D \cap T_2 = T_1 \cap T_2 = \langle (1, 1) \rangle$$

THE EXAMPLE

Let S be a group, and let $G = S_1 \times S_2$, where $S_1 \cong S_2 \cong S$.

Let $D = \{(x_1, x_2) \in G \mid x_1 = x_2\}, \quad T_1 = S_1 \times \langle 1 \rangle, \quad T_2 = \langle 1 \rangle \times S_2$.

Then $D \cong T_1 \cong T_2$, and these are pair-wise compliments:

$$\langle T_1, T_2 \rangle = \langle T_1, D \rangle = \langle D, T_2 \rangle = G$$

$$T_1 \cap D = D \cap T_2 = T_1 \cap T_2 = \langle (1, 1) \rangle$$

Let $\mathbf{A} = \langle G/T_1, G^{\mathbf{A}} \rangle$ = the algebra with universe the left cosets of T_1 in G, and basic operations the left multiplications by elements of G.

For each $g \in G$ the operation $g^{\mathbf{A}} \in G^{\mathbf{A}}$ is defined by

$$g^{\mathbf{A}}(xT_1) = (gx)T_1 \qquad (xT_1 \in G/T_1).$$

Define the algebra $\mathbf{C} = \langle G/T_2, G^{\mathbf{C}} \rangle$ similarly.

THE EXAMPLE

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

THE EXAMPLE

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2 x_1, g_1 x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

The example

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2 x_1, g_1 x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

THE EXAMPLE

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2x_1, g_1x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

It is easy to verify that this relation is a function, and in fact

$\varphi\colon \mathbf{A} \times \mathbf{C} \to \mathbf{B} \times \mathbf{C}$ is an isomorphism.

The Example

The algebra $\mathbf{B}$ will have universe $B = G/D$, but we define the action of G on B with a twist.

For each $g = (g_1, g_2) \in G$, for each $(x_1, x_2)D \in G/D$, define

$$g^{\mathbf{B}}((x_1, x_2)D) = (g_2x_1, g_1x_2)D.$$

Let $\mathbf{B} = \langle G/D, G^{\mathbf{B}} \rangle$, where $G^{\mathbf{B}} = \{g^{\mathbf{B}} \mid g \in G\}$.

Consider the binary relation $\varphi \subseteq (A \times C) \times (B \times C)$ that associates to each ordered pair

$$((x_1, x_2)T_1, (y_1, y_2)T_2) \in A \times C$$

the pair

$$((x_2, y_1)D, (y_1, y_2)T_2) \in B \times C$$

It is easy to verify that this relation is a function, and in fact

$$\varphi \colon \mathbf{A} \times \mathbf{C} \to \mathbf{B} \times \mathbf{C} \text{ is an isomorphism.}$$

Since φ leaves second coordinates fixed, $\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$.

CONCLUSION

Compare $\mathrm{Con}\,\mathbf{A}$ and $\mathrm{Con}\,\mathbf{B}$.

Conclusion

Compare $\operatorname{Con} \mathbf{A}$ and $\operatorname{Con} \mathbf{B}$.

$\operatorname{Con} \mathbf{A} \cong [\![T_1, G]\!] \leqslant \operatorname{Sub}(G)$, so $\operatorname{Con} \mathbf{A} \cong \operatorname{Sub}(S)$.

CONCLUSION

Compare $\operatorname{Con}\mathbf{A}$ and $\operatorname{Con}\mathbf{B}$.

$\operatorname{Con}\mathbf{A} \cong [\![T_1, G]\!] \leqslant \operatorname{Sub}(G)$, so $\operatorname{Con}\mathbf{A} \cong \operatorname{Sub}(S)$.

$\operatorname{Con}\mathbf{B}$ is isomorphic to the lattice of normal subgroups of S.

CONCLUSION

Compare $\operatorname{Con}\mathbf{A}$ and $\operatorname{Con}\mathbf{B}$.

$\operatorname{Con}\mathbf{A} \cong [\![T_1, G]\!] \leqslant \operatorname{Sub}(G)$, so $\operatorname{Con}\mathbf{A} \cong \operatorname{Sub}(S)$.

$\operatorname{Con}\mathbf{B}$ is isomorphic to the lattice of normal subgroups of S.

$$\operatorname{Con}\mathbf{B} \cong \operatorname{NSub}(S) \leqslant \operatorname{Sub}(S) \cong \operatorname{Con}\mathbf{A}$$

So, if S is any non-Dedekind group, $\operatorname{Con}\mathbf{B} \ncong \operatorname{Con}\mathbf{A}$.

Conclusion

Compare $\operatorname{Con}\mathbf{A}$ and $\operatorname{Con}\mathbf{B}$.

$\operatorname{Con}\mathbf{A} \cong [\![T_1, G]\!] \leqslant \operatorname{Sub}(G)$, so $\operatorname{Con}\mathbf{A} \cong \operatorname{Sub}(S)$.

$\operatorname{Con}\mathbf{B}$ is isomorphic to the lattice of normal subgroups of S.

$$\operatorname{Con}\mathbf{B} \cong \operatorname{NSub}(S) \leqslant \operatorname{Sub}(S) \cong \operatorname{Con}\mathbf{A}$$

So, if S is any non-Dedekind group, $\operatorname{Con}\mathbf{B} \ncong \operatorname{Con}\mathbf{A}$.

If S is a nonabelian simple group, then $\operatorname{Con}\mathbf{B} \cong \mathbf{2}$, while $\operatorname{Con}\mathbf{A} \cong \operatorname{Sub}(S)$ can be arbitrarily large.