Annals of Mathematical Logic 11 (1976) 193–259

THE DECISION PROBLEM FOR EQUATIONAL BASES OF ALGEBRAS *

George F. McNULTY
Department of Mathematics and Computer Science, University of South Carolina, Columbia, SC 29208, U.S.A.

Received 11 December 1973

0. Introduction

An *equational theory* (or more simply just *theory*) is the set of all equations holding universally in some algebra. A set B of equations is a *base* for the theory T provided B and T have the same models. A theory T is *base decidable* if and only if the set of finite bases of T is recursive. T is said to be *base undecidable* whenever it is not base decidable and T is *essentially base undecidable* just in the case that every theory based on any extension of T by finitely many equations (even allowing new operation symbols) is base undecidable. This paper is primarily concerned with exploring base decidability of equational theories. As consequences of the theorems proved here it turns out that almost every familiar finitely based equational theory is essentially base undecidable. As a very particular case we establish that the equational theory of Boolean algebras is essentially base undecidable, answering a question in [36]. The study of equational logic was essentially initiated by Birkhoff in [1] where a completeness theorem for equational logic is proved. Ref. [36] is a survey of equational logic prior to 1968 and is useful in placing our results in perspective.

It is a simple observation that every finitely based undecidable equational theory is base undecidable. The constructions of [18] and [31] of finitely presented semigroups whose word problems are each recur-

* This paper is for Bessie.

sively unsolvable provide the first examples of finitely based equational theories which are undecidable and hence base undecidable. Tarski [34] found a finitely based undecidable equational theory of relation algebras. Mal'cev [17] showed that there were finitely based undecidable equational theories in just two unary operation symbols and also that various finitely based theories of loops and quasigroups were undecidable. A finitely based undecidable theory of semigroups is presented in [25]. I do not know if any finitely based theory of groups is undecidable. Our theorems below yield many decidable theories that are essentially base undecidable. Perkins [28, 29] was the first to find a decidable equational theory which is base undecidable. He showed that the theory of the one element groupoid is essentially base undecidable. Perkins' terminology is different.

The principal results of this paper concerning base-decidability are:

0.1. If T is a finitely based theory such that there is a term θ and a variable x with $\theta \approx x \in T$ and some operation symbol of rank at least two or at least two distinct unary operation symbols occur in θ, then T is essentially base undecidable.

0.2. Fix a similarity type (equational language) provided with at least three unary operation symbols or some operation symbol of rank at least two. Every finitely based equational theory with a non-trivial model can be extended to a base undecidable theory in the same similarity type which also has a non-trivial model.

0.3. There is a base undecidable theory which is not essentially base undecidable, moreover this theory can be chosen in the language of groupoids: one binary operation symbol.

0.4. In the similarity type of two unary operation symbols there is exactly one finitely based equationally complete base decidable theory.

0.5. The equational theory of semilattices is base undecidable modulo the commutative law but base decidable modulo the associative law. (A theory T is *base decidable modulo an equation* if and only if the set of bases of T which include the equation is recursive.)

All of the results concerning base undecidability are proved by a uniform method of translating one theory into another. Although this translation is syntactical in character the proofs depend essentially on a model-theoretic device which, roughly speaking, descends from the construction of free algebras such as are typically used in the proof of the completeness theorem. This model-theoretic construction has proved useful and in fact originates in parts of (equational) logic having ostensibly little to do with decision problems. See the remarks at the end of Section 2. In this connection Theorems 2.9, 2.33, and 2.34, though technical in their statements, are major results of this paper and among those most likely to find other applications. Theorem 2.5 generalizes a result of Isbell [10] concerning Mycielski's universal terms. V.L. Murskii found 0.1 independently of me and at about the same time. His result is announced in [26] and a proof is sketched which is somewhat different from the one given here. In fact, I have been unable to use Murskii's methods to obtain the case with just two unary operation symbols.

Some of the results presented here were announded in [19–22].

Section 1 deals with notation and includes some well-known theorems from the literature that are used in later sections. The principal purpose of section 1 is to develop a notational system for equational logic. The section includes no new results. Section 2 contains the development of the major techniques used to establish the results stated above. In particular, the syntactical translations of one theory into another which were already mentioned, are studied in detail and for this purpose Jan Mycielski's notion of universal term is generalized. In section 3 the results of the previous section are used to establish theorems concerning base undecidable equational theories. Base decidable theories are the subject of Section 4 which also includes an example of a base undecidable theory which is not essentially base undecidable. Relevant open questions are gathered at the end of each of these sections. In Section 5 I take the opportunity to acknowledge the assistance many people have given to me connected with this paper.

1. Some fundamental notions from equational logic

This paper is written in the context of a set theory admitting proper classes as well as sets. *Ordinals* are conceived in such a way that each ordinal is the set of all smaller ordinals. *Cardinals* are initial ordinals. In particular, each natural number is a finite cardinal and 0 denotes at once the *empty set* and the least cardinal while ω denotes at once the set of natural numbers and the least infinite cardinal. If A is a set $|A|$ denotes the cardinality of A. If A and B are any classes ${}^AB$ is the class of all functions from A into B. "System" and "sequence" are each synonymous with function. If f is a function and a is in the domain of f then f_a, $f(a)$, and fa each denote the value of f at a. Whenever A is a subset of the domain of f, $f^*A = \{f_a : a \in A\}$. When A is the domain of the function f, then f is sometimes written as $\langle f_a : a \in A \rangle$ or $\langle f_a \rangle_{a \in A}$. If the domain of the function f is the natural number n, then f can also be written as $\langle f_0, ..., f_{n-1} \rangle$. Notice that the notion of ordered pair used to define the concept of function differs from the notion of two-termed sequence. Direct products of systems of sets are defined so as to be sets of functions. Consequently, the direct product of $\langle A, B \rangle$ is a set of two-termed sequences rather than a set of ordered pairs, for any sets A and B. If A is a system of sets PA denotes the *direct product* of A; if I is the domain of A, PA is sometimes written $P_{i \in I} A_i$. Q is an n-ary *operation* on the set A provided $Q \in {}^{({}^nA)}A$. Q is a *finitary operation* on A if Q is an n-ary operation on A for some $n \in \omega$. Unary operations on A, i.e., elements of ${}^{({}^1A)}A$, are generally identified with the naturally correlated functions in ${}^AA$, while a 0-ary is identified with the single element in its range. If Q is an n-ary operation on a non-empty set then n is said to be the rank of Q. ρ is the rank function: the domain of ρ is the class of operations on non-empty sets and if Q is in the domain of ρ then ρQ is the rank of Q.

Algebraic notions

An *algebra* $\mathfrak{A}$ is a two termed sequence $\langle A, F \rangle$ where A is a nonempty set and F is a system of finitary operations on A; A is the *universe* of $\mathfrak{A}$, the domain of F is the *index set* of $\mathfrak{A}$, and F is the system of *fundamental operations* of $\mathfrak{A}$. If $\mathfrak{A}$ is an algebra, Op $\mathfrak{A}$ denotes the system of fundamental operations of $\mathfrak{A}$; if Q is in the index set of $\mathfrak{A}$ the $Q^{\mathfrak{A}}$ and Op $\mathfrak{A}_Q$ both denote the corresponding operation of $\mathfrak{A}$. Algebras will usually be denoted by German capitals; their universes by the corresponding italic capitals.

The *similarity type* of the algebra $\mathfrak{A}$ is the system $\langle \rho(\text{Op}\,\mathfrak{A}_i) : i \in I \rangle$ where I is the index set of $\mathfrak{A}$, i.e. the similarity type of $\mathfrak{A}$ is the sequence of ranks of the fundamental operations in $\mathfrak{A}$. Two algebras are similar just in case they have the same similarity type. If $\mathfrak{A}$ is a system of similar algebras and the domain of $\mathfrak{A}$ is a set then $P\mathfrak{A}$ denotes the usual *direct product*; if the range of $\mathfrak{A}$ is $\{\mathfrak{B}\}$ and the domain of $\mathfrak{A}$ is I then $P\mathfrak{A}$ can be written as $\mathfrak{B}^I$. When $\mathfrak{A}$ is an algebra and $X \subseteq A$, then $S_g{}^{\mathfrak{A}}X$ denotes the *subuniverse of* $\mathfrak{A}$ *generated by* X and, provided $S_g{}^{\mathfrak{A}}X \neq 0$, $\mathfrak{S}_{\mathfrak{G}}{}^{\mathfrak{A}}X$ denotes the *subalgebra of* $\mathfrak{A}$ *generated by* X. $Pl\,\mathfrak{A}$ is the subalgebra of ${}^{\omega A}\mathfrak{A}$ generated by the set of projection functions. The universe of $Pl\,\mathfrak{A}$ is the *set of polynomials*, in ω variables over $\mathfrak{A}$.

For a more detailed development of these notions the reader is referred to [9, Chapter 0]. The notation adopted above differs only slightly from the notation in that book. A somewhat different notation is used in [7] which also includes an extensive bibliography of the general theory of algebras.

Equational logic

Associated with each similarity type is a first order language suitable for the expression of elementary properties of algebras with that similarity type. Of interest here is the associated equational language. This language is conceived as that fragment of the first order language which admits as formulas only universal sentences in prenex normal form whose quantifier free part is an equation between terms. Consequently, all connectives and quantifiers may be surpressed in equational languages. The development of equational logic sketched below follows closely the development Tarski presented in a course at Berkeley in 1968–69.

Equational languages are provided with three kinds of symbols: variable symbols, operation symbols, and a symbol for equality. Two equational languages differ only in operation symbols. The set of variable symbols is countably infinite but the set of operation symbols depends upon the similarity type and may be of any cardinality.

$\approx$ is the *equality symbol*. For each $i \in \omega$, v_i is the *i*th *variable* and $\mathrm{Va} = \{v_i : i \in \omega\}$ is the set of variables. If σ is a similarity type then the domain of σ is the set of *operation symbols*. All these symbols are taken to be distinct one termed sequences and $\approx$ and v_i, for each $i \in \omega$, are to be sets of finite rank (see Definition 1.14 below). The set of *expressions* of similarity type σ is just the set of all finite sequences generated by $\{\approx\} \cup \mathrm{Va} \cup$ domain of σ under concatenation. Juxtaposition of sequences (such as variables and operation symbols) represents concatenation.

Definition 1.0. Let σ be a similarity type. Te_σ, the set of *terms* of type σ, is the smallest set, X, such that

(i) $\mathrm{Va} \subseteq X$.

(ii) For each Q in the domain of σ and for each $\theta \in {}^{\sigma Q}X$

$$Q\theta_0 \ldots \theta_{\sigma Q - 1} \in X.$$

In particular, if $\sigma Q = 0$, then $Q \in \mathrm{Te}_\sigma$. 0-ary operation symbols are called *constants.* Terms result from the concatenation of finite sequences of variable symbols and operation symbols. Terms are uniquely readable in the sense no term can arise in this way from two different sequences of variable symbols and operation symbols. *Subterms* of a term are defined as usual.

Definition 1.1. Let σ be a similarity type and $\theta \in \mathrm{Te}_\sigma$.

(i) $\mathrm{V}\theta = \{i\colon v_i \text{ is a subterm of } \theta\}$,

(ii) $\mathrm{C}\theta = \{Q\colon Q \text{ is a subterm of } \theta \text{ and } \sigma Q = 0\}$,

(iii) $\mathrm{L}\theta$ is the number of occurrences of variable symbols and operation symbols in θ.

For any term θ, $\mathrm{V}\theta$ is the *variable support* of θ, $\mathrm{C}\theta$ is the *constant support* of θ, and $\mathrm{L}\theta$ is the *length* of θ.

Definition 1.2. Let σ be a similarity type. $\mathrm{Eq}_\sigma = \{\varphi \approx \psi\colon \varphi, \psi \in \mathrm{Te}_\sigma\}$.

Eq_σ is the *set of equations of type* σ. If ϵ is the equation $\varphi \approx \psi$, then ϵ_l is φ and ϵ_r is ψ.

Definition 1.3. Let σ be a similarity type and $\Gamma \subseteq \mathrm{Eq}_\sigma$. $t\Gamma = \{\epsilon_l\colon \epsilon \in \Gamma\} \cup \{\epsilon_r\colon \epsilon \in \Gamma\}$.

$t\Gamma$ is the set of all terms appearing as left or right sides of equations in Γ.

Definition 1.4. Let $\theta \in {}^\omega\mathrm{Te}_\sigma$ where σ is a similarity type. $\tau[\theta]$ is defined for all $\tau \in \mathrm{Te}_\sigma$ by recursion:

(i) $v_i[\theta] = \theta_i$ for $v_i \in \mathrm{Va}$

(ii) $Q\,\pi_0 \ldots \pi_{\sigma Q-1}[\theta] = Q\,\pi_0[\theta] \ldots \pi_{\sigma Q-1}[\theta]$ for all Q in the domain of σ and $\pi \in {}^{\sigma Q}\mathrm{Te}_\sigma$.

Strict use of this notation will be violated often since only a finite part of the sequence θ is needed to determine $\tau[\theta]$. $\tau[\theta]$ is called a *substitution instance* of τ.

Definition 1.5. Let φ, ψ and τ be terms.

$$\mathrm{R}(\varphi \approx \psi, \tau) = \{\delta\varphi\gamma\colon \delta\psi\gamma = \tau \text{ for some expressions } \delta \text{ and } \gamma\}$$
$$\cup \{\delta\psi\gamma\colon \delta\varphi\gamma = \tau \text{ for some expressions } \delta \text{ and } \gamma\}.$$

$\mathrm{R}(\varphi \approx \psi, \tau)$ is the set of terms that result from *replacement* in τ by means of $\varphi \approx \psi$.

Substitution and replacement can be used to describe rules of inference for equational logic. These rules will be formulated with the symbol $\vdash_\sigma$ and $\Sigma \vdash_\sigma \epsilon$ should read "ϵ is derivable from Σ". $\vdash_\sigma$ is a relation holding between sets of equations and single equations.

Definition 1.6. $\vdash_\sigma$ is the least relation R holding between subsets of Eq_σ and members of Eq_σ such that:

(i) $\Sigma R v_0 \approx v_0$ and $\Sigma R \epsilon$ for all $\Sigma \subseteq \mathrm{Eq}_\sigma$ and for all $\epsilon \in \Sigma$;

(ii) if $\Sigma R \varphi \approx \psi$ and $\theta \in {}^\omega\mathrm{Te}_\sigma$, then $\Sigma R \varphi[\theta] \approx \psi[\theta]$;

(iii) if $\Sigma R \varphi \approx \psi$, $\Sigma R \theta \approx \tau$ and $\delta \in R(\varphi \approx \psi, \theta)$, then $\Sigma R \tau \approx \delta$.

$0 \vdash_\sigma \epsilon$ is usually written as $\vdash_\sigma \epsilon$. $\mathrm{Ta}_\sigma = \{\epsilon : \vdash_\sigma \epsilon\}$ is the set of equational *tautologies* of similarity type σ. $\Sigma \vdash_\sigma \Delta$, where $\Delta \subseteq \mathrm{Eq}_\sigma$, means $\Sigma \vdash_\sigma \delta$ for all $\delta \in \Delta$. $\Sigma \nvdash_\sigma \delta$ means that $\Sigma \vdash_\sigma \delta$ does not hold.

Definition 1.7. Let σ be a similarity type.

(i) T is an *equational theory* (of similarity type σ) if and only if $T \subseteq \mathrm{Eq}_\sigma$ and for all $\epsilon \in \mathrm{Eq}_\sigma$ if $T \vdash_\sigma \epsilon$ then $\epsilon \in T$.

(ii) $\Theta[\Gamma]_\sigma = \{\epsilon : \Gamma \vdash_\sigma \epsilon\}$.

(iii) Γ is a *base* for T if and only if $\Theta[\Gamma]_\sigma = T$ for some similarity types σ such that $\Gamma \subseteq \mathrm{Eq}_\sigma$.

Another characterization of the notion of derivability proves useful, especially in proofs that require some kind of induction of derivations. This characterization provides a linear notion of derivation and limits the use of substitution.

Theorem 1.8. *Let σ be a similarity type and $\Sigma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_\sigma$. $\Sigma \vdash_\sigma \varphi \approx \psi$ if and only if for some $n \in \omega \sim \{0\}$ there is $\theta \in {}^n\mathrm{Te}_\sigma$ such that*

(i) *$\varphi = \theta_0$ and $\psi = \theta_{n-1}$;*

(ii) *for all $k < n - 1$ there is an equation $\pi \approx \tau \in \Sigma$ and $\eta \in {}^\omega\mathrm{Te}_\sigma$ so that $\theta_{k+1} \in \mathrm{R}(\pi[\eta] \approx \tau[\eta], \theta_k)$.*

The proof of this theorem follows by a straightforward induction argument. There are no unusual details in the proof so it is omitted. Notice, however, that if $\langle \theta_0, \ldots, \theta_{n-1} \rangle$ has the properties described above then $\Sigma \vdash_\sigma \theta_0 \approx \theta_k$ for each $k < n$. Such a sequence will be called a *derivation*.

Many concepts which apply to formal logical systems have natural formulations in equational logic. The following definition specifies some of these notions.

Definition 1.9. Let T be an equational theory of similarity type σ.

(i) T is *consistent* if and only if $T \neq \mathrm{Eq}_\sigma$.

(ii) T is *equationally complete* if and only if T is consistent and for all equational theories, Δ, such that $T \subseteq \Delta \subseteq \mathrm{Eq}_\sigma$, either $T = \Delta$ or $\Delta = \mathrm{Eq}_\sigma$.

(iii) T is *finitely based* if and only if T has a finite base.

(iv) A set $\Gamma \subseteq \mathrm{Eq}_\sigma$ is *irredundant* if and only if $\gamma \notin \Theta[\Gamma \sim \{\gamma\}]_\sigma$ for all $\gamma \in \Gamma$.

(v) $\nabla T = \{|\Gamma| : \Gamma$ is an irredundant base for $T\}$.

Certain relationships which hold between equational theories of different similarity types are central to the results and techniques of this paper. Two of the more prominent of these relationships are definitional equivalence and interpretability. Their treatment is based on the following definition.

Definition 1.10. Let σ and τ be similarity types where I is the domain of σ. Let $\delta \in {}^I\mathrm{Te}_\tau$ and $\epsilon \in {}^J\mathrm{Te}_\sigma$ where J is the domain of τ.

(i) δ is a *system of definitions* for σ in τ if and only if $\sigma Q = \mathrm{V}\delta_Q$ for each $Q \in I$.

(ii) δ is a *system of definitions* for σ in τ *in the wider sense* if and only if $\mathrm{V}\delta_Q \subseteq \sigma Q \cup \{0\}$ for all $Q \in I$.

(iii) If δ is a system of definitions for σ in τ in the wider sense, then in_δ, the *interpretation operator* on δ, is defined of Te_σ by recursion:

(a) $\mathrm{in}_\delta v_i = v_i$ for all $i \in \omega$

(b) $\mathrm{in}_\delta Q \approx \theta_0 \ldots \theta_{\sigma Q-1} = \delta_Q[\mathrm{in}_\delta \theta_0 \ldots \mathrm{in}_\delta \theta_{\sigma Q-1}]$ for $\theta \in {}^{\sigma Q}\mathrm{Te}_\sigma$ and $Q \in I$.

(iv) Let

$$\mathrm{Co}_\delta^\epsilon = \{\delta_Q \approx \delta_Q[v_1, v_1, \ldots] : Q \in I \text{ and } \sigma Q = 0\}$$
$$\cup \{\mathrm{in}_\delta \epsilon_P \approx Pv_0 \ldots v_{\sigma P-1} : P \in J\}.$$

If $\varphi \approx \psi \in \mathrm{Eq}_\sigma$, then $\mathrm{in}_\delta(\varphi \approx \psi)$ denotes $\mathrm{in}_\delta \varphi \approx \mathrm{in}_\delta \psi$.

Roughly, interpretation operators are the major tools used in this paper and the next section is devoted to their development. The notion of a system of definitions in the wider sense eases the definition of definitional equivalence. In particular, the condition of (ii) that $\mathrm{V}\delta_i \subseteq \sigma i \cup \{0\}$ is used rather than $\mathrm{V}\delta_i \subseteq \sigma i$ in order that constants may be defined by terms with variables.

Definition 1.11. Let Σ be an equational theory of similarity type σ and T be an equational theory of similarity type τ. $\Sigma \equiv_{\delta,\epsilon} T$ iff δ is a system of definitions in the wider sense for σ in τ and ϵ is a system of definitions in the wider sense for τ in σ and $\Theta[\mathrm{in}_\delta^* \Sigma \cup \mathrm{Co}_\delta^\epsilon]_\tau = T$ and $\Theta[\mathrm{in}_\epsilon^* T \cup \mathrm{Co}_\epsilon^\delta]_\sigma = \Sigma$. Σ and T are *definitionally equivalent* just in case there are ϵ and δ such that $\Sigma \equiv_{\delta,\epsilon} T$.

Definitional equivalence has been studied often. In the literature it is also called equational equivalence, polynomial equivalence, and even rational equivalence. It is elaborated in [6, 16, 36].

Equational logic inherits the concepts of *satisfaction* and *logical consequence* from first order logic. Whenever $\mathfrak{A}$ is an algebra of similarity type σ and $\tau \in \mathrm{Te}_\sigma$, $\tau^{\mathfrak{A}}$ denotes the polynomial on $\mathfrak{A}$ which is represented by τ. Note that the domain of $\tau^{\mathfrak{A}}$ is ${}^\omega A$. However, $\tau^{\mathfrak{A}}$ depends on only finitely many coordinates in ${}^\omega A$ so it is convenient to let $\tau^{\mathfrak{A}*}$ denote the function whose domain is ${}^{V\tau}A$ and for which if $a \in {}^\omega A$ then $\tau^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}*}(a \restriction V\tau)$. Let $\mathfrak{A}$ be an algebra and $\varphi \approx \psi$ an equation in the similarity type of $\mathfrak{A}$. $\mathfrak{A} \models \varphi \approx \psi$ iff $\varphi^{\mathfrak{A}} = \psi^{\mathfrak{A}}$. In this case $\mathfrak{A}$ is said to be a model of $\varphi \approx \psi$. This notation is also extended to sets of equations. For a set of equations, $\Sigma \cup \{\epsilon\}$, $\Sigma \models_\sigma \epsilon$ means that every model of Σ of similarity type σ is a model of ϵ; $\mathrm{Mo}_\sigma \Sigma$ denotes the class of models of Σ of similarity type σ. If K is a class of algebras of similarity type σ then $\mathrm{Th}\, K = \{\epsilon : \epsilon \in \mathrm{Eq}_\sigma$ and $\mathfrak{A} \models \epsilon$ for all $\mathfrak{A} \in K\}$. $\mathrm{Th}\, K$ is the equational theory of K. $\mathrm{Th}\{\mathfrak{A}\}$ is written $\mathrm{Th}\mathfrak{A}$. Notions originally defined for equational theories are applied to algebras and classes of algebras, e.g. algebras are said to be equationally complete or finitely based just in case their equational theories are.

The following theorem is a very strong completeness theorem established by Birkhoff [1].

Theorem 1.12. *Let $\Sigma \subseteq \mathrm{Eq}_\sigma$ for some similarity type σ. There is an algebra $\mathfrak{A}$ of type σ such that* $\mathrm{Th}\, \mathfrak{A} = \Theta[\Sigma]_\sigma$ *and if $\Theta[\Sigma]_\sigma$ is consistent then $\mathfrak{A}$ is generated by a countably infinite set and if $\Theta[\Sigma]_\sigma$ is inconsistent then $\mathfrak{A}$ has one element.*

Actually Birkhoff proved more. A full treatment of this result can be found in either [7] or [9]. We note the following corollary in passing.

Corollary 1.13. *Let K be a class of algebras of similarity type σ and $\Sigma \subseteq \mathrm{Eq}_\sigma$.*

(i) $\mathrm{Th}\,\mathrm{Mo}_\sigma \mathrm{Th}\, K = \mathrm{Th}\, K$

(ii) $\mathrm{Mo}_\sigma \mathrm{Th}\, \mathrm{Mo}_\sigma \Sigma = \mathrm{Mo}_\sigma \Sigma$

(iii) $\mathrm{Th}\,\mathrm{Mo}_\sigma \Sigma = \Theta[\Sigma]_\sigma$

(iv) $\mathrm{Th}\, K$ *is an equational theory.*

Most of the notation just described used subscripts to specify the similarity type. In practice, most of these subscripts will be suppressed, especially if there is only one similarity type at hand.

Recursive functions

The final portion of this section concerns the application of the notion of algorithm to equational logic. In fact, there is nothing particularly equational involved and the remarks apply equally well to a much wider class of systems. What is desired is to replace the intuitive idea of an algorithmic decision procedure by a precise mathematical definition. This is more usually done through the offices of some family of Gödel numberings, in our case one for each of an infinity of similarity types, which reduce the problem to finding an adequate notion of algorithm on ω. For ω it is generally agreed that the recursive functions correspond to the intuitive algorithms. What is suggested here is to forsake ω and to make the definition of recursive function with respect to the set of hereditarily finite sets instead. This yields the immediate advantage that one may speak of sets, sets of finite sets, sets of finite sequences, and so on as recursive or not recursive without recourse to any Gödel numbering. To this end each variable and the equality symbol was taken from the family of all hereditarily finite sets. In addition, there is a plentitude of operation symbols that are, in fact, hereditarily finite sets. Moreover, it is only necessary to define what a unary recursive function is because the set of hereditarily finite sets is closed with respect to finite direct products. Instead of proceeding entirely within the set of hereditarily finite sets with this definition, an explicit Gödel numbering of the hereditarily finite sets is given.

Definition 1.14. Let H be the smallest set X such that

(i) $0 \in X$;

(ii) If $A, B \in X$, then $\{A\} \in X$ and $A \cup B \in X$.

The sets in H are called the ***hereditarily finite sets*** or sets of finite rank. Evidently H includes ω. It is not difficult to show that $\in$ is a transitive relation of H, that if $K \cup L \subseteq H$ then $K \times L \subseteq H$, that all finite subsets of H are themselves members of H, or that all finite sequences of members of H are again members of H.

The next definition provides the Gödel numbering of H.

Definition 1.15. Let $F \in {}^{\omega}H$ such that

(i) $F0 = 0$

(ii) $F(2^n + j) = \{Fn\} \cup Fj$ for $n, j \in \omega$ and $j < 2^n$.

Verification of the next theorem is straightforward.

Theorem 1.16. *F is one-to-one and onto H.*

Definition 1.17. $f \in {}^H H$ is *recursive* if and only if $F^{-1} f F$ is a recursive function on ω, the set of natural numbers.

For an exposition of recursive functions on the natural numbers see Rogers [33]. All that is needed here is the theory of recursive functions of one variable on the natural numbers. In this connection, Julia Robinson has given a particularly nice formulation in [32]. Since F defined above is so simple, it is not difficult to see that Definition 1.17 supplies a formalization of the notion of algorithm exactly as adequate as that of recursive function on the natural numbers. A subset of H is recursive just in case its characteristic function is recursive. In this paper no functions or sets are shown to be recursive in all details. Such demonstrations would be extremely complex, though not essentially difficult. Instead, a more informal approach is taken: an algorithm is described for computing the desired function. From such a description, the process of actually constructing the recursive function will present no difficulties other than those attendant on the complexity of the algorithm.

Definition 1.18. Let σ be a similarity type. σ is *recursive* if and only if $\sigma \subseteq H$ and σ has a recursive domain and σ is a recursive set.

Now we make one more stipulation: Va is a recursive set. Suppose σ is recursive. Apparently Te_σ, Eq_σ, and $\{\Gamma : |\Gamma| < \omega$ and $\Gamma \subseteq \mathrm{Eq}_\sigma\}$ are all recursive subsets of H. If $\Sigma \subseteq \mathrm{Eq}_\sigma$ and Σ is recursively enumerable then $\Theta[\Sigma]_\sigma$ is also recursively enumerable.

For another formalization of the notion of recursive functions over the family of sets of finite rank see Platek [30].

2. How to build jointly universal sets of terms and use them to reduce one similarity type to another.

Given two different similarity types σ and τ it is natural to ask whether everything expressible in σ is also expressible in τ in such a way that the notion of logical consequence is preserved. More precisely whether there is a function $F \in {}^{\mathrm{Eq}_\sigma}\mathrm{Eq}_\tau$ that is one-to-one and such that if $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$ then $\Sigma \vdash_\sigma \epsilon$ iff $F^*\Sigma \vdash_\tau F\epsilon$. One of the goals of this section is to show that many such functions exist between almost any two similarity types. (The only restrictions necessary are those regarding cardinalities and the fact that operations of rank less than two cannot be used exclusively to construct a polynomial depending on more variables.) In fact all such functions constructed here turn out to be natural extensions of the interpretation operators defined in the previous section and some have even stronger and more convenient syntactical properties. The first part of this section provides an analysis of when interpretation operators can be used in this way. According to Definition 1.10, the interpretation operator in_δ is completely determined by δ. The analysis below provides two sufficient conditions on the range of δ – that it is jointly universal or that it satisfy the subterm condition – under which in_δ reduces σ to τ as described above. The property of being jointly universal has a model theoretic character while the stronger subtern condition is purely syntactical. After a brief discussion on how to relativize these notions – particularly in connection with the commutative and associative laws – some of their fundamental properties are developed. Section 2 concludes with constructions of infinite sets of terms having one of these two properties and such that each element of the set retains some nice logical properties of a fixed predetermined term. For example, suppose θ is a term composed from the variable x and a binary operation symbol. Then there is an infinite set, Δ, of terms in the binary operation symbol and the variable x so that Δ satisfies the above-mentioned syntactic condition and $\{\theta \approx x\} \vdash \{\rho \approx x : \rho \in \Delta\}$.

It turns out that the existence of such sets of terms and the possibility of constructing them from a single nearly arbitrary term have applications beyond the scope of this paper. For this reason, this section is more substantial than is necessary to prove the theorems mentioned in the introduction. Further application of these results are given in [23, 24].

Definition 2.0. Let σ and τ be similarity types and let δ be a system of definitions for σ in τ. in_δ is a *reduction* of σ to τ if and only if for all $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$, $\Sigma \vdash_\sigma \epsilon$ just in case $\mathrm{in}_\delta^* \Sigma \vdash_\tau \mathrm{in}_\delta\, \epsilon$.

Remark 2.1. It is true that for any interpretation operator, in_δ, and any $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$ if $\Sigma \vdash_\sigma \epsilon$ then $\mathrm{in}_\delta^* \Sigma \vdash_\tau \mathrm{in}_\delta\, \epsilon$, as can be easily established either proof theoretically or by means of models. Consequently, to show that in_δ is a reduction it is only necessary to show that if $\mathrm{in}_\delta^* \Sigma \vdash_\tau \mathrm{in}_\delta \epsilon$ then $\Sigma \vdash_\sigma \epsilon$, for every $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$. If in_δ is a reduction it is easy to see that δ is one-to-one. Recall that whenever δ is a system of definitions for σ in τ and i is in the domain of σ then $\sigma i = \mathrm{V}\delta i$, i.e. δ *preserves rank.*

The condition "if $\mathrm{in}_\delta^* \Sigma \vdash_\tau \mathrm{in}_\delta \epsilon$ then $\Sigma \vdash_\sigma \epsilon$ for any $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$" insists that $\mathrm{in}_\delta^* \Sigma$ have enough models to invalidate $\mathrm{in}_\delta \epsilon$ whenever $\Sigma \nvdash_\sigma \epsilon$. One way to accomplish this is to provide a nice way to convert every algebra, $\mathfrak{A}$, with ω generators into an algebra $\mathfrak{B}$ so that $\mathfrak{A} \models \epsilon$ iff $\mathfrak{B} \models \mathrm{in}_\delta \epsilon$ for each $\epsilon \in \mathrm{Eq}_\sigma$. This is the point of the next definition. Say that an assignment of finitary functions over some set to a set Δ of terms *agrees according to rank* provided that whenever $\theta \in \Delta$ has exactly n distinct variables then the function assigned to θ is n-ary.

Definition 2.2. Let κ be a cardinal. Δ is *jointly κ universal* if and only if Δ is a set of terms and for any assignment f of finitary functions over κ to Δ which agrees according to rank there is an algebra $\mathfrak{A}$ such that $\theta^{\mathfrak{A}^*} = f\theta$ for every $\theta \in \Delta$.

The prototypical example of a jointly universal set of terms is $\{Qv_0 \ldots v_{\sigma Q-1} : Q$ is in the domain of $\sigma\}$ for any similarity type σ. This fits well with the intuition that the terms assigned to the "operation symbols" by a reduction should behave like operation symbols at least with respect to algebras with ω generators. For this reason "Δ is a set of generalized operation symbols" would be a better phrase than "Δ is jointly $|\sigma| + \omega$ universal". The second phrase is adopted here for several reasons: (A) It reveals the dependence on the cardinal κ; (B) It is not yet known whether range δ is jointly $|\sigma| + \omega$ universal whenever in_δ is a reduction of σ to τ; (C) The second phrase extends already established terminology. In fact Jan Mycielski calls a term, θ, κ universal if $\{\theta\}$ is jointly κ universal. He raised questions about the existence of terms universal in some cardinals but not in others and especially the question whether $\{\theta : \theta \in \mathrm{Te}_\sigma$ and θ is κ universal for each $\kappa \in S\}$ is recursive for various recursive similarity types σ and various classes S of cardinals. The problem remains open, even when σ gives just two operation symbols both of which are unary and S is just $\{\omega\}$. This particular instance of the problem, together with some related material, is dealt with in [10]. Some results below generalize theorems of Isbell.

Theorem 2.3. (*The reduction theorem.*) *Let σ and τ be similarity types* *and let δ be a system of definitions for σ in τ such that δ is one-to-one and the range of δ is jointly $|\sigma| + \omega$ universal.* in_δ *is a reduction of σ to τ.*

Proof. By Remark 2.1 it is only necessary to establish $\mathrm{in}_\delta^* \Sigma \vdash_\tau \mathrm{in}_\delta \epsilon$ implies $\Sigma \vdash_\sigma \epsilon$ for every $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$. Fix $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$ such that $\Sigma \not\vdash_\sigma \epsilon$. By Theorem 1.12 Σ has a model $\mathfrak{A}$ of cardinality $|\sigma| + \omega$ and $\mathfrak{A} \not\models \epsilon$. Assign to elements of the range of δ operations of $\mathfrak{A}$ in the natural way: for Q in the domain of σ assign to δ_Q the operation $Q^{\mathfrak{A}}$. Since δ is one-to-one every element of the range of δ is assigned exactly one operation of Since δ is a system of definitions, this assignment agrees according to rank. Since the range of δ is jointly $|\sigma| + \omega$ universal there is an algebra $\mathfrak{B}$ so that for Q in domain σ, $Q^{\mathfrak{A}} = \delta_Q^{\mathfrak{B}}$. A simple induction on terms establishes $\varphi^{\mathfrak{A}} = (\mathrm{in}_\delta\, \varphi)^{\mathfrak{B}}$ for each term φ of type σ. The theorem follows immediately.

We have no algorithm for checking whether a given set is jointly universal, not even an immediate way to build jointly universal sets. This is a reason why Mycielski's question concerning the recursiveness of $\{\theta: \theta \in \mathrm{Te}_\sigma$ and θ is κ-universal$\}$ is interesting. The remainder of this section is devoted to providing partial remedies for this situation.

Various jointly universal sets of terms have appeared in the literature. Some historical remarks are included at the end of this section. However, here it should be noted that Ralph McKenzie was, to my knowledge, first to formulate a nice syntactic condition on sets of terms sufficient to insure that they be jointly universal. In fact, he established a version of the reduction theorem and a weaker version of Theorem 2.5.

Definition 2.4. [McKenzie] Δ satisfies the *subterm condition* if and only if Δ is a set of terms, none of which are variables, such that if $\delta, \theta \in \Delta$ and γ is a non-variable subterm of δ such that θ has a substitution instance identical with a substitution instance of γ then $\theta = \delta = \gamma$.

What the subterm condition guarantees is that in evaluating a term θ from a set Δ satisfying the subterm condition, all the proper subterms of θ can be evaluated just as they are in the absolutely free algebra without affecting the value of θ or any other member of Δ. This fact is reflected in the proof of the following theorem.

Theorem 2.5. *Let σ be a similarity type. If Δ is a set of terms of type σ which satisfies the subterm condition, φ, ψ, $\pi \in \mathrm{Te}_\sigma$ such that $\varphi \neq \psi$ and*

$\mathrm{L}\varphi$, $\mathrm{L}\psi < \mathrm{L}\delta$ for every $\delta \in \Delta$ and F is an assignment of finitary functions over $|\sigma| + \omega$ to Δ that agrees according to rank then there is an algebra, $\mathfrak{A}$, such that $A = |\sigma| + \omega$ and

(i) *$\delta^{\mathfrak{A}*} = F\delta$ for each $\delta \in \Delta$;*

(ii) *$\mathfrak{A} \not\models \varphi \approx \psi$;*

(iii) *If $F\delta(a) = 0$ for each $\delta \in \Delta$ and $a \in {}^{\mathrm{V}\delta}A$ such that $0 \in$ range of a then $\mathfrak{A} \models \varphi \approx \pi$ only if $\mathrm{V}\varphi \supseteq \mathrm{V}\pi$.*

Proof. There are two cases. First suppose (iii) holds vacuously. It is enough to discover an algebra $\mathfrak{A}$ with universe Te_σ that satisfies (i) and (ii) where F is construed as an assignment over Te_σ to Δ that agrees according to rank. To this end let Q be in the domain of σ and $\theta \in {}^{\sigma Q}\mathrm{Te}_\sigma$ and define $Q^{\mathfrak{A}}(\theta)$ as follows:

$$Q^{\mathfrak{A}}(\theta_0, \ldots, \theta_{\sigma Q-1}) = \begin{cases} F\delta(\eta) & \text{if } Q\theta_0 \ldots \theta_{\sigma Q-1} = \delta[\eta] \\ & \text{for some } \delta \in \Delta \text{ and some } \eta \in {}^{\mathrm{V}\delta}\mathrm{Te}_\sigma; \\ Q\theta_0 \ldots \theta_{\sigma Q-1} & \text{otherwise.} \end{cases}$$

In order to see that $Q^{\mathfrak{A}}$ is well defined suppose that $Q\theta_0 \ldots \theta_{\sigma Q-1} = \delta[\eta] = \delta'[\eta']$ for $\delta, \delta' \in \Delta$ and $\eta \in {}^{\mathrm{V}\delta}\mathrm{Te}_\sigma$ and $\eta' \in {}^{\mathrm{V}\delta'}\mathrm{Te}_\sigma$. Since Δ satisfies the subterm condition it follows that $\delta = \delta'$ and furthermore $\eta = \eta'$. Hence $Q^{\mathfrak{A}}$ is well defined.

Claim 1. If $\delta \in \Delta$ and η is a proper subterm of δ then $\eta^{\mathfrak{A}}(\theta) = \eta[\theta]$ for each $\theta \in {}^{\omega}\mathrm{Te}_\sigma$.

Proof. Proceed by induction on η. Suppose η is v_j for some $j \in \omega$. Clearly $v_j^{\mathfrak{A}}(\theta) = \theta_j = v_j[\theta]$. If η is $Q\gamma_0 \ldots \gamma_{\sigma Q-1}$ for some $\gamma \in {}^{\sigma Q}\mathrm{Te}_\sigma$ then

$$\eta^{\mathfrak{A}}(\theta) = Q^{\mathfrak{A}}(\gamma_0^{\mathfrak{A}}(\theta), \ldots, \gamma_{\sigma Q-1}^{\mathfrak{A}}(\theta)) = Q^{\mathfrak{A}}(\gamma_0[\theta], \ldots, \gamma_{\sigma Q-1}[\theta])$$

by induction hypothesis. It follows now by the subterm condition on Δ that

$$\eta^{\mathfrak{A}}(\theta) = Q^{\mathfrak{A}}(\gamma_0[\theta], \ldots, \gamma_{\sigma Q-1}[\theta]) = Q\gamma_0[\theta] \ldots \gamma_{\sigma Q-1}[\theta].$$

Therefore

$$\eta^{\mathfrak{A}}(\theta) = (Q\gamma_0 \ldots \gamma_{\sigma Q-1})[\theta] = \eta[\theta]$$

and the claim is finished.

Claim 2. $\delta^{\mathfrak{A}*} = F\delta$ for each $\delta \in \Delta$.

Proof. Suppose

$$\delta = Q\eta_0 \ldots \eta_{\sigma Q-1} \quad \text{and} \quad \theta \in {}^{\omega}\mathrm{Te}_{\sigma} .$$

Then

$$\delta^{\mathfrak{A}}(\theta) = Q^{\mathfrak{A}}(\eta_0(\theta), \ldots, \eta^{\mathfrak{A}}_{\sigma Q-1}(\theta)) = Q^{\mathfrak{A}}(\eta_0[\theta], \ldots, \eta_{\sigma Q-1}[\theta])$$

by Claim 1. So $\delta^{\mathfrak{A}}(\theta) = F\delta(\theta \restriction \mathrm{V}\delta)$, by the definition of $Q^{\mathfrak{A}}$. Hence $\delta^{\mathfrak{A}^*} = F\delta$ establishing the claim and property (i) of the theorem.

Claim 3. $\mathfrak{A} \not\models \varphi \approx \psi$.

Proof. Let $\theta = \langle v_j : j \in \omega \rangle$. Proceed by induction on φ to show that $\varphi^{\mathfrak{A}}(\theta) = \varphi$. If φ is v_k for some $k \in \omega$ then $\varphi^{\mathfrak{A}}(\theta) = v_k$ is clear. If φ is $Q\eta_0 \ldots \eta_{\sigma Q-1}$ for some $\eta \in {}^{\sigma Q}\mathrm{Te}_{\sigma}$ then $\varphi^{\mathfrak{A}}(\theta) = Q^{\mathfrak{A}}(\eta_0(\theta), \ldots, \eta_{\sigma Q-1}(\theta))$. Now $\mathrm{L}\eta_k < \mathrm{L}\varphi < \delta$ for each $\delta \in \Delta$ and so by inductive hypothesis $\eta_k(\theta) = \eta_k$, for each $k < Q$.
$\mathfrak{A}$ It follows that $\varphi^{\mathfrak{A}}(\theta) = Q^{\mathfrak{A}}(\eta_0, \ldots, \eta_{\sigma Q-1})$. Consequently, $\varphi^{\mathfrak{A}}(\theta) = \varphi$ by the definition of $Q^{\mathfrak{A}}$ and the induction is complete. The same induction argument shows $\psi^{\mathfrak{A}}(\theta) = \psi$. Since $\varphi \neq \psi$ it follows that $\mathfrak{A} \not\models \varphi \approx \psi$ and the claim and property (ii) of the theorem are established.

Now suppose property (iii) of the theorem does not hold vacuously, i.e. $F\delta(a) = 0$ for each $\delta \in \Delta$ and $a \in {}^{\mathrm{V}\delta}|\sigma| + \omega$ such that 0 is in the range of a. It is enough to discover an algebra, $\mathfrak{A}$, with universe $\mathrm{Te}_{\sigma} \cup \{0\}$ that satisfies (i), (ii), and (iii) where F is construed as an assignment over $\mathrm{Te}_{\sigma} \cup \{0\}$ to Δ that agrees according to rank and such that (iii) is not vacuous. Again let Q be in the domain of σ and $\theta \in {}^{\sigma Q}(\mathrm{Te}_{\sigma} \cup \{0\})$ and define $Q^{\mathfrak{A}}(\theta)$ as follows:

$$Q^{\mathfrak{A}}(\theta) = 0 \quad \text{if} \quad \theta_j = 0 \quad \text{for some } j \in \sigma Q$$

otherwise define $Q^{\mathfrak{A}}$ just as in the first case.

It is routine to establish Claims 1, 2, and 3 for this new definition of $Q^{\mathfrak{A}}$ – noting in Claim 2 that $F\delta(\theta) = 0$ when $\theta \in {}^{\mathrm{V}\delta}(\mathrm{Te}_{\sigma} \cup \{0\})$ such that 0 is in the range of θ.

Claim 4. $\mathfrak{A} \models \varphi \approx \pi$ only if $\mathrm{V}\varphi \supseteq \mathrm{V}\pi$.

Proof. Suppose $\mathfrak{A} \models \varphi \approx \pi$. Let $\theta \in {}^{\omega}(\mathrm{Te}_{\sigma} \cup \{0\})$ such that

$$\theta_j = \begin{cases} v_j & \text{if } j \in \mathrm{V}\varphi , \\ 0 & \text{otherwise} . \end{cases}$$

Then $\varphi^{\mathfrak{A}}(\theta) = \varphi$, just as in Claim 3. Consequently $\pi^{\mathfrak{A}}(\theta) \neq 0$ and therefore $V\varphi \supseteq V\pi$, completing the proof of Claim 4.

Theorem 2.5 is established in all particulars.

Corollary 2.6. *Let σ be a similarity type. If $\Delta \subseteq \mathrm{Te}_\sigma$ and Δ satisfies the subterm condition then Δ is jointly κ universal for each $\kappa > |\sigma| + \omega$.*

Corollary 2.7. *Let σ and τ be similarity types and δ be a one-to-one system of definitions for σ in τ such that the range of δ satisfies the subterm condition.* in_δ *is a reduction of σ to τ.*

Proof. Observe that the range of δ has cardinality $|\sigma|$ and so there is $\tau' \subseteq \tau$ so that $\mathrm{Te}_{\tau'} \supseteq$ range of δ and $|\tau'| + \omega = |\sigma| + \omega$. Apply Corollary 2.6 to Theorem 2.3.

Notational Remark. Whenever Q is an expression and n is a natural number Q^n denotes Q concatenated with itself n times. Q^0 is the empty expression and $Q^{n+1} = Q^n Q$ for $n \in \omega$.

Example 2.8. Let f and g be distinct unary operation symbols. $\{f^2 g^{n+1} fgv_0 : n \in \omega\}$ satisfies the subterm condition.

Proof. Let θ and φ be any terms and suppose ψ is a non-variable subterm of $f^2 g^{n+1} fgv_0$ and $\psi[\varphi] = f^2 g^{m+1} fg[\theta]$. ψ falls into one of the cases below.

(i) $\psi = gv_0$. This is impossible since $g \neq f$.

(ii) $\psi = fgv_0$. This is impossible since $fg \neq f^2$.

(iii) $\psi = g^k fgv_0$ where $0 < k \leq n+1$. This is impossible since $g \neq f$.

(iv) $\psi = fg^{n+1} fgv_0$. This is impossible since $fg \neq f^2$.

(v) $\psi = f^2 g^{n+1} fgv_0$. This is possible only if $n = m$.

Consequently $\{f^2 g^{n+1} fgv_0 : n \in \omega\}$ satisfies the subterm condition.

Whenever σ is said to *have κ operation symbols of rank n* just in case $\kappa = |\{Q\colon Q$ is in the domain of σ and $\sigma Q = n\}|$.

The next theorem establishes the existence of sets of terms which are, in some sense, maximal with respect to the subterm condition.

Theorem 2.9. (The existence theorem). *Let σ be a similarity type.*

(i) *If $\sigma = 0$ then the only subset of Te_σ satisfying the subterm condition is the empty set.*

(ii) *If σ has only operations of rank* 0 *then* $\Delta \subseteq \mathrm{Te}_\sigma$ *satisfies the subterm condition if and only if no variables occur in any member of* Δ.

(iii) *If σ has one unary operation symbol and no other operation symbol then* $\Delta \subseteq \mathrm{Te}_\sigma$ *satisfies the subterm condition if and only if* $\Delta \cap \mathrm{Va} = 0$ *and the operation symbol occurs no more than once in* Δ.

(iv) *If* $\sigma = \sigma' \cup \sigma''$ *and* σ' *has only operation symbols of rank* 0 *and* σ'' *has exactly one operation symbol and that one unary then* $\Delta \subseteq \mathrm{Te}_\sigma$ *satisfies the subterm condition if and only if no constant symbol occurs in more than one member of* Δ *and if* $\theta \in \Delta$ *and a variable occurs in* θ *then the unary operation symbol occurs exactly once in* θ *and in no other member of* Δ.

(v) *If σ has an operation symbol of rank at least two or at least two unary operation symbols then there is* $\Delta \subseteq \mathrm{Te}_\sigma$ *such that*

$$|\Delta \cap \{\theta : \mathrm{V}\theta = n \text{ and } \theta \in \mathrm{Te}_\sigma\}| =$$
$$= |\{\theta : \mathrm{V}\theta = n \text{ and } \theta \in \mathrm{Te}_\sigma\}|$$

for each $n \in \omega$ *and* Δ *satisfies the subterm condition.*

Proof. (i)–(iv) are immediate from the definition of the subterm condition. By means of a construction similar to Example 2.8, (v) follows unless σ has an operation symbol of rank at least two. For the sake of simplicity assume that σ has a binary operation symbol and denote it by Q. The construction given below adapts easily to the case of operation symbols of greater rank.

To begin the construction let $\varphi_k = Qv_0Q^{k+2}v_0^{k+3}$ and $\psi_j = Q^jv_0 \dots v_j$, for every $j, k \in \omega$. Finally, let $\theta_{j,k} = \varphi_0[\psi_j[\varphi_{k+1}[v_0], \dots, \varphi_{k+1}[v_j]]]$ and let $\Delta_0 = \{\theta_{j,k} : j, k \in \omega\}$.

Claim 1. $\{\varphi_k : k \in \omega\}$ satisfies the subterm condition.

Proof. Suppose π is a non-variable subterm of φ_k, η and γ are terms and $\pi[\eta] = \varphi_{k'}[\gamma]$. Now π is either φ_k itself or else $Q^{n+1}v_0^{n+2}$ for some $n < k+2$. In the first case it follows easily that $\eta = \gamma$ and so $k = k'$ and $\pi = \varphi_k = \varphi_{k'}$. The second case is impossible since then $Q^n\eta^{n+1} = \gamma$ and $\eta = Q^{k'+2}\gamma^{k'+3}$. So $\{\varphi_k : k \in \omega\}$ satisfies the subterm condition.

Claim 2. Δ_0 satisfies the subterm condition.

Proof. Suppose π is a non-variable subterm of $\theta_{j,k}$, $\eta, \gamma \in {}^\omega\mathrm{Te}$, and

$\pi[\eta]^{\frown} = \theta_{j',k'}[\gamma]$. π is limited to

Case 1. π is a non-variable subterm of $\varphi_{k+1}[v_n]$ for some $n \leqslant j$. This is impossible since $\theta_{j',k'}$ is a substitution instance of φ_0 and $\{\varphi_k : k \in \omega\}$ satisfies the subterm condition by Claim 1.

Case 2. $\pi = Q^n \varphi_{k+1}[v_0] \dots \varphi_{k+1}[v_n]$ for some n, $0 < n \leqslant j$. Let $\xi = \langle \varphi_{k'+1}[\gamma_i] : i \in \omega \rangle$. Then $\theta_{j',k'}[\gamma] = Q\psi_{j'}[\xi] Q\psi_{j'}[\xi]\psi_{j'}[\xi]$. It follows that $\varphi_{k+1}[\eta_n] = Q\psi_{j'}[\xi]\ \psi_{j'}[\xi]$ and this is impossible by the definition of φ_{k+1}.

Case 3. $\pi = Q\psi_j[\varphi_{k+1}[v_0], \dots, \varphi_{k+1}[v_j]]\,\psi_j[\varphi_{k+1}[v_0], \dots, \varphi_{k+1}[v_j]]$. This is impossible since $\{\varphi_0\}$ satisfies the subterm condition by Claim 1.

Case 4. $\pi = \theta_{j,k}$. Then $\psi_j[\varphi_{k+1}[\eta_0], \dots, \varphi_{k+1}[\eta_j]] = \psi_{j'}[\varphi_{k'+1}[\gamma_0], \dots, \varphi_{k'+1}[\gamma_{j'}]]$. So $\varphi_{k+1}[\eta_j] = \varphi_{k'+1}[\gamma_{j'}]$ and by Claim 1 $k = k'$. Evidently $j = j'$ and so $\theta_{j,k} = \theta_{j',k'}$.

Consequently Δ_0 satisfies the subterm condition and Claim 2 is established.

Let π_R be $Rv_0 v_0 \dots v_0$ for each R in the domain of σ. For $j, k \in \omega$ and R in the domain of σ let $\theta_{j,k,R}$ be $\theta_{j,k}[v_0, v_1, \dots, v_{j-1}, \pi_R]$. Finally let $\theta_{j,k,R,P}$ be $\theta_{j,k,R}[v_0, v_1, \dots, v_{j-2}, \pi_P]$ for each $j, k \in \omega$ and each R, P in the domain of σ. Let $\Delta = \{\theta_{j,k,R,P} : j, k \in \omega$ and R, P in the domain of σ such that $R \neq Q \neq P\}$. It is easy to see that Δ satisfies the subterm condition since it has already been shown that Δ_0 satisfies the subterm condition and Δ was obtained from Δ_0 by some simple substitutions. Δ also has the required cardinality properties since

$$V\theta_{j,k,R,P} = \begin{cases} j-2 & \text{if } j > 2, \\ 1 & \text{if } \sigma n > 0, \quad j = 1 \\ 1 & \text{if } \sigma n > 0, \quad j = 0 \text{ and } \sigma R > 0 \\ 0 & \text{if } \sigma n = 0, \quad j < 2 \\ 0 & \text{if } \sigma i = 0, \quad j = 0\,. \end{cases}$$

Hence the proof of the theorem is complete.

The proof of Theorem 2.9 was carried out in such detail in order to demonstrate how the subterm condition may be established. Subsequently, the demonstrations that various sets of terms satisfy the subterm condition will be less detailed. Theorem 2.9 leads to the following definition.

Definition 2.10. (i) A similarity type σ is *trivial* if and only if σ has at most one unary operation symbol and no operation symbol of rank more than one.

(ii) A term θ is *trivial* if and only if $\theta \in \mathrm{Te}_\sigma$ for some trivial similarity type σ.

Corollary 2.11. *Let σ and τ be similarity types. If $|\sigma| < |\tau| + \omega$, τ is non-trivial, σ has a constant symbol only if τ has a constant symbol, and σ has an operation symbol of rank more than one only if τ does, then there is a system of definitions δ for σ in τ such that* in_δ *is a reduction of σ to τ.*

Remark 2.12. It should be noticed that a more elaborate notion of reduction is possible that eliminates the necessity to be concerned about constant symbols in this corollary. Recalling Definition 1.10 "δ is a system of definitions for σ in τ" could be changed to mean $V\delta_Q = \sigma Q$ for all Q in the domain of δ such that $\sigma Q > 0$ and $V\delta_Q \subseteq \{0\}$ for all Q in the domain of σ such that $\sigma Q = 0$. in_δ could then be called a reduction of σ to τ provided $\Sigma \vdash_\sigma \epsilon$ iff $\mathrm{in}_\delta^* \Sigma \cup \mathrm{Co}_\delta \vdash_\tau \mathrm{in}_\delta \epsilon$. This broader notion of reduction does not find application in this paper but does give rise to some uninteresting complications in some of the proofs presented here.

It is natural to wonder if the concepts of reduction, joint universality, and the subterm condition can be relativized to equational classes different from the class of all algebras of some similarity type. This is the subject of the following digression.

Definition 2.13. Let σ and τ be similarity types, δ be a system of definitions for σ in τ, $\Delta \subseteq \mathrm{Te}_\tau$, $\Gamma \subseteq \mathrm{Eq}_\tau$ and κ be a cardinal.

(i) in_δ is a *reduction of σ to τ modulo* Γ if and only if $\Sigma \vdash \epsilon$ just in case $\mathrm{in}_\delta^* \Sigma \cup \Gamma \vdash \mathrm{in}_\delta \epsilon$, for all $\Sigma \cup \{\epsilon\} \subseteq \mathrm{Eq}_\sigma$.

(ii) Δ is *jointly κ universal modulo* Γ if and only if for every assignment f of functions over κ to Δ that agrees according to rank there is an algebra $\mathfrak{A}$ with universe κ such that $\mathfrak{A} \models \Gamma$ and $\theta^{\mathfrak{A}^*} = f\theta$ for every $\theta \in \Delta$.

(iii) Δ satisfies the *subterm condition modulo* Γ if and only if

(a) $\Gamma \nvdash \theta[\eta] \approx \rho$ for any $\theta \in \Delta$, $\eta \in {}^\omega\mathrm{Te}_\tau$ where ρ is any variable or any substitution instance of a proper non-variable subterm of any member of $t\Gamma$;

(b) if $\theta, \varphi \in \Delta$, $\pi, \eta \in {}^\omega\mathrm{Te}_\tau$, and γ is any nonvariable subterm of θ such that $\Gamma \vdash \gamma[\pi] \approx \varphi[\eta]$, then $\theta = \gamma = \varphi$ and $\Gamma \vdash \pi_i \approx \eta_i$ for every $i \in V\theta$.

Analogs to the reduction theorem and Theorem 2.5 can be established for these notions. Apparently, the existence of diverse sets of terms satisfying the subterm condition modulo Γ would make it possible to find models of Γ possessing "reducts" with similarly diverse properties. For example, if there were a term θ in which the variables v_0 and v_1 occur and such that $\{\theta\}$ is jointly ω universal modulo Γ then Γ would have a countable model $\mathfrak{A}$ such that if $\mathfrak{A}$ is expanded to $\mathfrak{A}'$ by adjoining all finitary polynomials over $\mathfrak{A}$ as new operation symbols and Σ is any set of equations in any countable similarity type then $\mathfrak{A}'$ has a reduct which is a model of Σ, up to some permutation of operations of $\mathfrak{A}'$. This property is too strong to expect it to hold for many sets of equations – especially those which arise most commonly. It cannot happen, for example, for any set Γ of equations such that $\Theta[\Gamma]$ has only countably many equationally complete extensions. This will be demonstrated in [24]. Among the equational theories which have only countably many equationally complete extensions occur all equational theories of semigroups, groups, rings, lattices, and Boolean algebras. What is demonstrated below is that there is no term, θ, (even in one variable) and no cardinal $\kappa > 1$ such that $\{\theta\}$ is jointly κ universal modulo the associative law. In a more positive vein an analog for the existence theorem is established modulo the commutative law.

As a matter of convenience terms and polynomials will be written in the most familiar manner: the convention of writing operations on the left is dropped, momentarily, and a binary operation symbol $\cdot$ is introduced and terms (similarly polynomials) are defined in such a way that $\varphi \cdot \psi$ is the term resulting from applying the operation symbol to φ and ψ. For the next two theorems, the similarity type has $\cdot$ as its only operation symbol.

Theorem 2.14. *For any term θ and any cardinal $\kappa > 1$, $\{\theta\}$ is not jointly κ universal modulo the associative law.*

Proof. It is only necessary to consider terms in one variable. Suppose x is the variable occurring in θ. Hence there is $n > 0$ so that

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \vdash \theta \approx x^n .$$

There are two cases according to whether $\kappa > 2$ or not.

Case 1. $\kappa = 2$. Let $f \in {}^2 2$ such that $f_0 = 1$ and $f_1 = 0$. Let $\langle 2, \circ \rangle$ be any two element semigroup and suppose $0^n = 1$ and $1^n = 0$. Now $1 \circ 1 = 0$ for otherwise $1^n = 1$. Similarly $0 \circ 0 = 1$. Therefore $1 \circ 0 = 1 \circ (1 \circ 1) = (1 \circ 1) \circ 1 = 0 \circ 1$ and so $(1 \circ 0) \circ (1 \circ 0) = (0 \circ 0) \circ (1 \circ 1) = 1 \circ 0$. But this means that $1 \circ 0$ can't be either 0 or 1 – a contradiction. It follows that the assignment of f to θ can't work in any two element semigroup.

Case 2. $\kappa > 2$. This case, though slightly more complicated, can be handled in a similar manner. Its proof is omitted.

Theorem 2.15. *There is a set* Δ *of terms which satisfies the subterm condition modulo the commutative law such that* $|\Delta \cap \{\theta : V\theta = n+1$ *and* θ *is a term*$\}| = \omega$, *for every* $n \in \omega$.

Proof. The proof follows closely that a case (v) of the existence theorem. Let

$$\varphi_k = (v_0 \cdot v_0) \cdot \underbrace{(v_0 \cdot (v_0 \cdot \ldots (v_0 \cdot v_0)) \ldots)}_{k+3\ v_0\text{'s}} \qquad \text{for every } k \in \omega$$

and let

$$\psi_j = v_0 \cdot (v_1 \cdot \ldots (v_{j-1} \cdot v_j) \ldots) \qquad \text{for every } j \in \omega.$$

Again let

$$\theta_{j,k} = \varphi_0[\psi_j[\varphi_{k+1}[v_0], \ldots, \varphi_{k+1}[v_j]]] \qquad \text{for every } j, k \in \omega.$$

Finally let

$$\Delta = \{\theta_{j,k} : j, k \in \omega\} .$$

It is enough to show that Δ satisfies the subterm condition modulo the commutative law. We leave this to the reader.

Before returning to the major task of this section, some compactness type notions will be discussed. A set Δ of terms of similarity type τ is a reduction set if and only if there is a similarity type σ and a system δ of definitions for σ in τ such that Δ is the range of δ and in_δ is a reduction of σ to τ. David Kelly pointed out to me that it is an easy consequence of the compactness theorem that a set of terms is a reduction set just in case every finite subset of it is also a reduction set. It is even clearer that a set of terms satisfies the subterm condition if and only if every subset with no more than two members satisfies the subterm condition. It is therefore surprising that compactness properties for jointly universal sets of terms are largely unknown and perhaps rare.

Theorem 2.16. *Let $0 < k < \omega$. If Δ is any set of terms such that every finite subset of Δ is jointly k universal then Δ is jointly k universal.*

Proof. Suppose Δ is not jointly k universal. Then there is an assignment f of finitary functions over k to Δ which agrees according to rank and yet such that for any algebra $\mathfrak{A}$ with universe k there is $\theta \in \Delta$ so that $f_\theta \neq \theta^{\mathfrak{A}*}$. Let $= \langle k; 0, ..., k-1, f_\theta \rangle_{\theta \in \Delta}$. Define Γ to be

$$\{\forall \bar{x}\, [\theta \approx f_\theta \bar{x}] : \theta \in \Delta\} \cup \{\forall x\, [\bigvee_{i \in k} x \approx i]\}$$

together with the first order theory of $\mathfrak{B}$. Then Γ must be inconsistent. So by the compactness theorem there is a finite subset, Δ_0, of Δ so that

$$\{\forall \bar{x}\, [\theta \approx f_\theta \bar{x}] : \theta \in \Delta_0\} \cup \{\forall x\, [\bigvee_{i \in k} x \approx 1]\}$$

together with the first-order theory of $\langle k; 0, ..., k-1, f_\theta \rangle_{\theta \in \Delta_0}$ inconsistent. Therefore Δ_0 is not jointly k universal and the theorem is established.

Corollary 2.17. *Δ is jointly universal in every finite cardinal if and only if every finite subset of Δ is jointly universal in every finite cardinal.*

The remainder of this section is devoted to the construction of infinite, jointly universal sets of terms each of which retain some convenient properties of a fixed though arbitrary non-trivial term. The next result is a corollary of the definitions involved.

Corollary 2.18. *Suppose Δ_0 and Δ_1 are disjoint sets of terms and that $\theta \in {}^\omega\Delta_1$ and θ is one-to-one. Let κ be a cardinal*

(i) *If $\Delta_0 \cup \Delta_1$ is jointly κ universal then $\{\delta[\theta] : \delta \in \Delta_0\}$ is jointly κ universal.*

(ii) *If $\Delta_0 \cup \Delta_1$ satisfies the subterm condition then $\{\delta[\theta] : \delta \in \Delta_0\}$ satisfies the subterm condition.*

A similar result holds for interpretation operators as well.

Theorem 2.19. *Let σ and τ be similarity types and δ be a system of definitions for σ in τ such that δ is one-to-one and the range of δ is jointly $|\sigma| + \omega$ universal. If $\Sigma \subseteq \mathrm{Te}_\sigma$ and Σ is jointly $|\sigma| + \omega$ universal then $\mathrm{in}_\delta^*\Sigma$ is jointly $|\sigma| + \omega$ universal.*

Proof. Let F be an assignment of finitary functions over $|\sigma| + \omega$ to $\text{in}_\delta^* \Sigma$ that agrees according to rank. in_δ is one-to-one so define G, an assignment of finitary functions over $|\sigma| + \omega$ to Σ so that $G_\theta = F_{\text{in}_\delta \theta}$ for $\theta \in \Sigma$. Since δ is a system of definitions G agrees according to rank. Since Σ is jointly $|\sigma| + \omega$ universal there is an algebra with universe $|\sigma| + \omega$ so that $\theta^{\mathfrak{A}^*} = G_\theta = F_{\text{in}_\delta \theta}$ for each $\theta \in \Sigma$. Since δ is a system of definitions and the range of δ is jointly $|\sigma| + \omega$ universal there is an algebra $\mathfrak{B}$ with universe $|\sigma| + \omega$ such that $\delta_Q^{\mathfrak{B}^*} = Q^{\mathfrak{A}}$ for every Q in the domain of σ. A simple induction on terms yields $(\text{in}_\delta \varphi)^{\mathfrak{B}} = \varphi^{\mathfrak{A}}$ for every $\varphi \in \text{Te}_\sigma$. In particular $(\text{in}_\delta\ \theta)^{\mathfrak{B}^*} = \theta^{\mathfrak{A}^*} = G_\theta = F_{\text{in}_\delta \theta}$ for each $\theta \in \Sigma$ and therefore $\text{in}_\delta^* \Sigma$ is jointly $|\sigma| + \omega$ universal.

In order to establish the corresponding result for the subterm condition the following definition and lemmas prove useful.

Definition 2.20. Let σ and τ be similarity types and δ be a system of definitions for σ in τ. θ is *δ-simple* if and only if $\theta \in \text{Te}_\tau$ and $\theta \neq \text{in}_\delta \varphi[\psi]$ for any $\varphi \in \text{Te}_\sigma \sim \text{Va}$ and $\psi \in {}^\omega\text{Te}_\tau$. $S_\delta \text{Te}_\tau$ denotes the *set of δ-simple terms* of type τ.

Lemma 2.21. *Let σ and τ be similarity types and δ be a one-to-one system of definitions for σ in τ such that the range of δ satisfies the subterm condition. For any $n \in \omega$, $\theta, \pi \in {}^n\text{Te}_\sigma$ and $\varphi, \psi \in {}^\omega S_\delta \text{Te}_\tau$ if $\text{in}_\delta \theta_i[\varphi] = \text{in}_\delta \pi_i[\psi]$ for each $i \in n$ then there are $\varphi', \psi' \in {}^\omega\text{Te}_\sigma$ such that $\theta_i[\varphi'] = \pi_i[\psi']$ for each $i \in n$.*

Proof. The proof is by induction on $m = \max(\{L\theta_i : i \in n\} \cup \{L\pi_i : i \in n\})$. $m = 1$. In this case $\{\theta_i : i \in n\} \cup \{\pi_i : i \in n\}$ consists exclusively of variables and constants. Since δ is a one-to-one system of definitions and φ and ψ are sequences of δ-simple terms then by letting $\varphi' = \psi' = \langle v_0 : i \in \omega\rangle$ the theorem holds.

Inductive step. Let $q > 1$ and assume the theorem is true whenever $m < q$. For any $j \in n$, $L\theta_j > 1$ there is Q_j in the domain of σ and $\gamma \in {}^{\sigma Q_j}\text{Te}_\sigma$ such that $\theta_j = Q_j \gamma_0 \cdots \gamma_{\sigma Q_j - 1}$. Now $\text{in}_\delta \theta_j[\varphi] = \text{in}_\delta \pi_j[\psi]$, in_δ is one-to-one, and φ and ψ are sequences of δ-simple terms. Consequently, there is $\eta \in {}^{\sigma Q_j}\text{Te}_\sigma$ such that $\pi_j = Q_j \eta_0 \cdots \eta_{\sigma Q_j - 1}$ and $\text{in}_\delta \gamma_i[\varphi] = \text{in}_\delta \eta_i[\psi]$ for each $i \in \sigma Q_j$. So for each $j \in n$ and $i \in \sigma Q_j$ let $\hat{\theta}_{j,i} = \gamma_i$ where θ_j is $Q_j \gamma_0 \cdots \gamma_{\sigma Q_j - 1}$ and $\hat{\pi}_{j,i} = \eta_i$ where π_j is $Q_j \eta_0 \cdots \eta_{\sigma Q_j - 1}$. By the inductive hypothesis there are $\varphi', \psi' \in {}^\omega\text{Te}_\sigma$ such that $\theta_j[\varphi'] = \pi_j[\psi']$ if $L\theta_j = 1$ and $\hat{\theta}_{j,i}[\varphi'] = \hat{\pi}_{j,i}[\psi']$ if

$L\theta_j > 1$ and $i \in \sigma Q_j$. But if $L\theta_j > 1$ then $\theta_j[\varphi'] = Q_j\hat{\theta}_{j,0}[\varphi'] \dots \hat{\theta}_{j,\sigma Q_j-1}[\varphi'] = Q_j\hat{\pi}_{j,0}[\psi'] \dots \hat{\pi}_{j,\sigma Q_j-1}[\psi'] = \pi_j[\psi']$ and the induction is complete.

Lemma 2.22. *Let σ and τ be similarity types and δ be a system of definitions for σ in τ such that δ is one-to-one and the range of δ satisfies the subterm condition. Let φ be any non-variable term of type σ and θ be any nonvariable subterm of $\mathrm{in}_\delta \varphi$.*

(i) *There are Q in the domain of σ, γ a non-variable subterm of δ_Q and $\psi \in {}^\omega\mathrm{Te}_\tau$ such that $\theta = \gamma[\psi]$.*

(ii) *For every γ in the range of δ such that θ is a substitution instance of γ there is φ', a subterm of φ, such that $\theta = \mathrm{in}_\delta \varphi'$.*

Proof. Let $\varphi = P\pi_0 \dots \pi_{\sigma P-1}$. The proof proceeds by induction on the number of proper non-variable subterms of φ.

(i) φ has no proper non-variable subterms. Then $\pi_0, \dots, \pi_{\sigma P-1} \in \mathrm{Va}$ and $\mathrm{in}_\delta \varphi = \delta_P[\pi_0, \dots, \pi_{\sigma P-1}]$. Since θ is not a variable there is a non-variable subterm, γ, of δ_P and $\theta = \gamma[\pi_0, \dots, \pi_{\sigma P-1}]$. If there is γ' in the range of δ and θ is a substitution instance of γ' then $\gamma' = \gamma = \delta_P$ by the subterm condition and so $\theta = \mathrm{in}_\delta \varphi$.

(ii) (Inductive step). Assume that every non-variable term with fewer proper non-variable subterms than φ satisfies the theorem. θ is a subterm of $\mathrm{in}_\delta \varphi = \delta_P[\mathrm{in}_\delta \pi_0, \dots, \mathrm{in}_\delta \pi_{\sigma P-1}]$. So either there is a non-variable subterm γ of δ_P such that $\theta = \gamma[\mathrm{in}_\delta \pi_0, \dots, \mathrm{in}_\delta \pi_{\sigma P-1}]$ or else θ is a subterm of $\mathrm{in}_\delta \pi_k$ for some $k \in \sigma P$. In the first case if γ' is in the range of δ and θ is a substitution instance of γ' then $\gamma' = \gamma = \delta_P$ by the subterm condition. So $\theta = \mathrm{in}_\delta \varphi$. In the second case the induction hypothesis applies so there is R in the domain of σ and γ, a non-variable subterm of δ_R such that θ is a substitution instance of γ. So (i) holds. Since π_k is a subterm of φ it follows that (ii) holds as well. This establishes the lemma.

Theorem 2.23. *Let σ and τ be similarity types and δ be a system of definitions for σ in τ such that δ is one-to-one and the range of δ satisfies then subterm condition. If $\Sigma \subseteq \mathrm{Te}_\sigma$ and Σ satisfies the subterm condition then $\mathrm{in}_\delta^* \Sigma$ satisfies the subterm condition.*

Proof. Whenever $\theta \in \mathrm{Te}_\tau$ there are $\varphi \in \mathrm{Te}_\sigma$ and $\psi \in {}^\omega S_\delta \mathrm{Te}_\tau$ such that $\theta = \mathrm{in}_\delta \varphi[\psi]$. This may be established by induction on θ. Let $\theta, \pi \in \Sigma$, η be a non-variable subterm of $\mathrm{in}_\delta \theta$, and $\varphi, \psi \in {}^\omega\mathrm{Te}_\tau$ such that $\eta[\varphi] = \mathrm{in}_\delta \pi[\psi]$. By Lemma 2.22 there is θ', a non-variable subterm of θ such that $\eta = \mathrm{in}_\delta\,\theta'$. Hence $\mathrm{in}_\delta\,\theta'[\varphi] = \mathrm{in}_\delta\,\pi[\psi]$. There are $\varphi', \psi' \in {}^\omega S_\delta\,\mathrm{Te}_\tau$ and $\varphi'', \psi'' \in {}^\omega\mathrm{Te}_\sigma$ such that $\varphi_i = \mathrm{in}_\delta\,\varphi''[\varphi']$ and $\psi_i = \mathrm{in}_\delta\,\psi''[\psi']$ for each $i \in \omega$. Consequently $\mathrm{in}_\delta\,(\theta'[\varphi''])[\varphi'] = \mathrm{in}_\delta\,(\pi[\psi''])[\psi']$. By Lemma 2.21 there are $\overline{\varphi}, \overline{\psi} \in {}^\omega\mathrm{Te}_\sigma$ such that $(\theta'[\varphi''])[\overline{\varphi}] = (\pi[\psi''])[\overline{\psi}]$. By the subterm condition on Σ, $\theta' = \theta = \pi$. So $\eta = \mathrm{in}_\delta\,\theta' = \mathrm{in}_\delta\,\theta = \mathrm{in}_\delta\,\pi$. Therefore $\mathrm{in}_\delta^*\Sigma$ satisfies the subterm condition.

The next definition introduces the notion of absorption, a kind of idempotence in equational logic, which turns out to be a highly useful property of terms.

Definition 2.24. Δ *absorbs* Σ *for* Γ if and only if Δ and Σ are sets of terms, Γ is a set of equations and $\Gamma \vdash \{\theta[\delta, \delta, \delta, \ldots] \approx \delta \colon \delta \in \Delta \text{ and } \theta \in \Sigma\}$.

If δ and θ are terms and Γ is a set of equations then "δ absorbs θ for Γ" usually replaces "$\{\delta\}$ absorbs $\{\theta\}$ for Γ". The next definition is admittedly artificial but it seems to be the most convenient way to formulate Theorems 2.26 and 2.30 as well as their consequences throughout the rest of the paper.

Definition 2.25.[1]

(i) If $\theta = f^{m+1}gHf^n v_i$, where f and g are any two distinct unary operation symbols, H is a (possibly empty) string of unary operation symbols such that the rightmost symbol is not f, and $m, n, i \in \omega$, then let $m(\theta) = \{f^{m+1} gHv_i, f^m gHv_i\}$.

(ii) If $\theta = HQ\varphi_0 \ldots \varphi_{r-1}$, where Q is an operation symbol of rank $r > 1$, H is a string of unary operation symbols and $\varphi_0, \ldots, \varphi_{r-1}$ are terms, then $m(\theta) = \{\theta, \varphi_0, \ldots, \varphi_{r-1}\}$.

Theorem 2.26. *Let θ be a non-trivial term in which all operation symbols occurring are unary and $V\theta \neq 0$. There is a set Σ of terms in the same operation symbols as θ such that*

(i) *Σ is infinite,*

(ii) *Σ satisfies the subterm condition,*

(iii) *If $\Delta \cup m(\theta)$ absorbs θ for Γ then $\Delta \cup \Sigma$ absorbs Σ for Γ and $\Gamma \vdash \{\varphi \approx \psi \colon \varphi, \psi \in \Sigma\}$.*

[1] This definition differs from that of $\theta\#$ in McNulty [21] in some details, though not in conception. I don't know if the earlier definition is adequate for its intended purpose.

Proof. The proof amounts to a series of constructions beginning with θ and ending with Σ. At each stage the terms developed are closer to satisfying the subterm condition. Care is taken at each stage to preserve the absorption properties of θ. Since θ is non-trivial it has the form $f^{m+1}gHf^n v_i$ for some distinct unary operations f and g, some (possibly empty) string of unary operations H not ending in f, and some $m,n,i \in \omega$. So without loss of generality assume

$$\theta = f^{m+1}gHf^n v_0 \ .$$

Then

$$m(\theta) = \{f^{m+1}gHv_0, f^m gHv_0\} \ .$$

Let

$$\pi_k = f^{m+k+1}gHf^{n-k}v_0$$

for each $k \in n+1$.

Let

$$\varphi_j = (f^{n+1})^{j+1}(f^m gH)^{j+1}v_0$$

for $j \in \omega$. Let $p-1$ be the number of times f occurs in H. Finally, let $\psi_0 = \varphi_{2p}$ and $\psi_1 = \varphi_p[\varphi_p]$. The idea is to show that $\{\psi_0, \psi_1\}$ satisfies (ii) and (iii). So suppose $\Delta \cup m(\theta)$ absorbs θ for Γ. Observe the following:

(a) $\pi_0 = \theta$.

(b) π_k absorbs π_{k+1} for Γ whenever $k \in n$.
Proof. $\pi_{k+1}[\pi_k] = f^{m+k+2}gHf^{n-(k+1)}f^{m+k+1}gHf^{n-k}v_0 = f^{k+1}\theta[f^m gHf^{n-k}v_0$
So

$$\Gamma \vdash \pi_{k+1}[\pi_k] \approx f^{m+k+1}gHf^{n-k}v_0,$$

since $f^m gHv_0$ absorbs θ for Γ.

(c) π_k absorbs π_k for Γ whenever $k \in n+1$.
Proof. $\pi_k[\pi_k] = f^{m+k+1}gHf^{n-k}f^{m+k+1}gHf^{n-k}v_0 = f^k\theta[f^{m+1}gHf^{n-k}v_0]$.
So

$$\Gamma \vdash \pi_k[\pi_k] \approx f^{m+k+1}gHf^{n-k}v_0$$

since $f^{m+1}gHv_0$ absorbs θ for Γ.

(d) $\Delta \cup m(\theta) \cup \{\pi_0, ..., \pi_n\}$ absorbs π_n for Γ.
Proof. Suppose $k \in n$. By (b) $\Gamma \vdash \pi_k \approx \pi_{k+1}[\pi_k]$. Hence

$$\Gamma \vdash \pi_k \approx \pi_n[\pi_{n-1}[\dots \pi_{k+1}[\pi_k] \dots] \ .$$

Similarly

$$\Gamma \vdash \pi_k \approx \pi_{n-1}[\pi_{n-2}[\dots \pi_{k+1}[\pi_k]\dots]].$$

So $\Gamma \vdash \pi_k \approx \pi_n[\pi_k]$. By (c) $\Gamma \vdash \pi_n \approx \pi_n[\pi_n]$. Let $\delta \in \Delta \cup m(\theta)$. Observe that $\Gamma \vdash \pi_0 \approx \pi_n[\pi_0]$ and therefore $\Gamma \vdash \pi_0[\delta] \approx \pi_n]\pi_0[\delta]]$. Consequently $\Gamma \vdash \delta \approx \pi_n[\delta]$.

(e) $\varphi_0 = \pi_n$.

(f) $\Gamma \vdash \varphi_j \approx \varphi_0$ for all $j \in \omega$.

Proof. By induction on j. If $j = 0$ this is immediate. Suppose $j > 0$ and $\Gamma \vdash \varphi_{j-1} \approx \varphi_0$. Now

$$\begin{aligned}\varphi_j &= (f^{n+1})^{j+1}(f^m gH)^{j+1} v_0\\ &= (f^{n+1})^j f^{n+1} f^m gH(f^m gH)^j v_0\\ &= (f^{\ +1})^j \pi_n f^m gH(f^m gH)^{j-1} v_0 .\end{aligned}$$

So

$$\Gamma \vdash \varphi_j \approx (f^{n+1})^j (f^m gH)^j v_0$$

since $\Gamma \vdash \pi_n [f^m gHv_0] \approx f^m gHv_0$ by (d). Consequently $\Gamma \vdash \varphi_j \approx \varphi_{j-1}$ and by the inductive assumption $\Gamma \varphi_j \approx \varphi_0$.

(g) $\Gamma \vdash \psi_0 \approx \psi_1$ and, in fact, $\Gamma \vdash \{\psi_0 \approx \varphi_0, \psi_1 \approx \varphi_0\}$.

Proof. By (f) $\Gamma \vdash \psi_0 \approx \varphi_0$ and $\Gamma \vdash \psi_1 \approx \varphi_0[\varphi_0]$, but $\Gamma \vdash \varphi_0[\varphi_0] \approx \varphi_0$ by (c) and (e). Consequently $\Gamma \vdash \psi_1 \approx \varphi_0$.

(h) $\Delta \cup \{\psi_0, \psi_1\}$ absorbs $\{\psi_0, \psi_1\}$ for Γ.

Proof. Suppose $\delta \in \Delta$. By (d) and (e) $\Gamma \vdash \varphi_0[\delta] \approx \delta$ and so by (g) $\Gamma \vdash \{\psi_0[\delta] \approx \delta, \psi_1[\delta] \approx \delta\}$. Furthermore (g) yields $\Gamma \vdash \{\psi_0[\psi_1] \approx \varphi_0, \psi_1[\psi_1] \approx \varphi_0, \psi_1[\psi_0] \approx \varphi_0, \psi_0[\psi_0] \approx \varphi_0\}$ so $\{\psi_0, \psi_1\}$ absorbs $\{\psi_0, \psi_1\}$ for Γ.

(i) $\{\psi_0, \psi_1\}$ satisfies the subterm condition.

Proof. Notice that

$$\psi_0 = f^{(2p+1)(n+1)+m} gH(f^m gH)^{2p} v_0$$

while

$$\psi_1 = f^{(p+1)(n+1)+m} gH(f^m gH)^p f^{(p+1)(n+1)+m} gH(f^m gH)^p v_0 .$$

Recall that f occurs only $p-1$ times in H and that H does not end in f. It is only a matter of inspection to see that $\{\psi_0, \psi_1\}$ satisfies the subterm condition.

To complete the proof of the theorem let σ be a similarity type whose only operation symbols are the unary operation symbols f and g. Let τ be any type in which θ is a term. Let η be the system of definitions for σ in τ such that $\text{in}_\eta fv_0 = \psi_0$ and $\text{in}_\eta gv_0 = \psi_1$. η is one-to-one and the range of η satisfies the subterm condition. Let $\Sigma_0 = \{f^2g^{n+1}fgv_0 : n \in \omega\}$ – the set shown in Example 2.8 to satisfy the subterm condition. Finally, let $\Sigma = \text{in}_\eta^* \Sigma_0$. By Theorem 2.23 Σ satisfies the subterm condition. Since in_η is one-to-one and Σ_0 is infinite Σ must also be infinite. Now suppose $\Delta \cup m(\theta)$ absorbs θ for Γ. By (h) above $\Delta \cup \{\psi_0, \psi_1\}$ absorbs $\{\psi_0, \psi_1\}$ for Γ so $\Delta \cup \Sigma$ must absorb Σ for Γ, by the definition of Σ and $\Gamma \vdash \{\gamma \approx \varphi_0 : \gamma \in \Sigma\}$. This completes the proof of the theorem.

Now suppose $\theta = Q\varphi_0 \ldots \varphi_{r-1}$ where Q is an operation symbol of rank $r > 1$. The next immediate goal is to establish the analog of Theorem 2.26 for terms of this type. In order to accomplish this the following definition and lemmas prove useful. When θ is a term $\theta^{[0]} = v_0$ and $\theta^{[k+1]} = \theta^{[k]}[\theta, \theta, \theta, \ldots]$ for $k \in \omega$.

Definition 2.27. Let $\theta = Q\varphi_0 \ldots \varphi_{r-1}$ be a term so that Q is an operation symbol of rank r and $\varphi_0, \ldots, \varphi_{r-1}$ are terms. ψ is *the associate of θ of type η* if and only if $\eta \in {}^r\omega$ and $\psi = Q\theta^{[\eta_0]}[\varphi_0, \varphi_0, \ldots] \ldots \theta^{[\eta_{r-1}]}[\varphi_{r-1}, \varphi_{r-1}, \ldots]$.

Lemma 2.28. *If θ is a non-variable term and ψ is an associate of θ and γ is a non-variable subterm of ψ and $\gamma \neq \psi$ then γ is a substitution instance of some non-variable subterm of θ.*

Proof. Suppose $\theta = Q\varphi_0 \ldots \varphi_{r-1}$ for an operation symbol Q and terms $\varphi_0, \ldots, \varphi_{r-1}$ and that $\psi = Q\theta^{[\eta_0]}[\varphi_0, \varphi_0, \ldots] \ldots \theta^{[\eta_{r-1}]}[\varphi_{r-1}, \varphi_{r-1}, \ldots]$ where $\eta \in {}^r\omega$. Then γ is a subterm of $\theta^{[\eta_j]}[\varphi_j, \varphi_j, \ldots]$, where $j \in r$. Either γ is a non-variable subterm of φ_j or there is a non-variable subterm, π, of $\theta^{[\eta_j]}$ and $\gamma = \pi[\varphi_j, \varphi_j, \ldots]$, in the first case γ is a non-variable subterm of θ since φ_j is. In the second case it is easy to establish by induction on η_j that π is a substitution instance of some non-variable subterm of θ. In this way the lemma is proved.

By Lemma 2.28, it is possible to ignore all proper subterms of associates of a term, θ, unless they are already subterms of θ itself, when checking for the subterm condition on a set of associates of θ. On the other hand, there are infinitely many associates of any non-variable, nonconstant term and most of them will be quite complex. As is shown below, many sets of associates will turn out to satisfy the subterm condition, provided the original term begins with an operation symbol of rank at least two. Evidently, the associates of a term have the absorption properties required in Theorem 2.26. If η is a sequence of natural numbers and j is in the domain of η then $\eta : j$ denotes the sequence with the same domain so that

$$\eta_k = \begin{cases} \eta_k & \text{if } j \neq k, \\ \eta_k + 1 & \text{if } j = k, \end{cases} \qquad \text{for } k \text{ in the domain of } \eta.$$

Lemma 2.29. *Let $\theta = Q\varphi_0 \ldots \varphi_{r-1}$ where Q is an operation symbol of rank $r > 1$ and $\varphi_0, \ldots, \varphi_{r-1}$ are terms and x is the only variable occurring in θ. There is $\eta \in {}^r\omega$ such that $\{\psi_j : \psi_j$ is the associate of θ of type $\eta : j$ and $j \in r\}$ satisfies the subterm condition.*

Proof. In order to establish the subterm condition it is necessary to examine substitution instances of subterms (of associates) of θ. Let $\eta \in {}^r\omega$ and ψ be the associate of θ of type η. Let γ be any non-variable subterm of ψ and let π, α be terms such that

$$\psi[\pi] = \gamma[\alpha]\,. \qquad ☆$$

Since Q is the first symbol of ψ and γ is not a variable, Q must be the first symbol of γ. Let $\gamma = Q\xi_0 \ldots \xi_{r-1}$ for the terms $\xi_0, \ldots, \xi_{r-1}$. From ☆ it follows for each $i \in r$ that $V\theta^{[\eta_i]}[\varphi_i[\pi]] = V\xi_i[\alpha]$ and $C\theta^{[\eta_i]}[\varphi_i[\pi]] = C\xi_i[\alpha]$. Consequently for each $i \in r$

$$|V\theta^{[\eta_i]}| \cdot |V\varphi_i| \cdot |V\pi| = |V\xi_i| \cdot |V\alpha| \tag{1}$$

and

$$|C\theta^{[\eta_i]}[\varphi_i]| + |V\theta^{[\eta_i]}[\varphi_i]| \cdot |C\pi| = |C\xi_i| + |V\xi_i| \cdot |C\alpha|\,. \tag{2}$$

In order to establish the lemma a sequence, $\eta \in {}^r\omega$, must be found so that ☆ fails unless $\psi = \gamma$. There are two cases depending on whether $C\theta$ is empty or not.

Case I. $C\theta = 0$. By Lemma 2.28 γ can be limited to ψ and the non-variable subterms of θ in any consideration of whether ☆ holds for some terms π and α. Furthermore, if ☆ holds for some terms π and α then it holds for terms in which no constants occur. So for any γ which is a non-variable subterm of θ and for which ☆ holds it follows from (1) that

$$\frac{|V\theta^{[\eta_0]}|}{|V\theta^{[\eta_1]}|} = \frac{|V\xi_0|}{|V\xi_1|} \cdot \frac{|V\varphi_1|}{|V\varphi_0|} . \tag{3}$$

As γ ranges through the non-variable subterms of θ, the right side of (3) takes on only finitely many values. So let $\eta_1 = 0$ and pick η_0 so large that

$$|V\theta^{[\eta_0]}| > |V\theta| \cdot \left[\frac{|V\xi_0|}{|V\xi_1|} \cdot \frac{|V\varphi_1|}{|V\varphi_0|}\right] \tag{4}$$

η is arbitrary otherwise. Let $j \in r$ and γ be a non-variable subterm of θ. Let ψ be the associate of θ of type $\eta : j$. ☆ holds for no terms π and α. (Otherwise (4) would fail.) Suppose $j, k \in r$ and let ψ_j be the associate of θ of type $\eta : j$ and ψ_k be the associate of type $\eta : k$. Let π and α be terms such that $\psi_j[\pi] = \psi_k[\alpha]$. Then $\theta^{[\eta_j+1]}[\varphi_j[\pi]] = \theta^{[\eta_j]}[\varphi_j[\alpha]]$ and $\theta^{[\eta_k+1]}[\varphi_k[\alpha]] = \theta^{[\eta_k]}[\varphi_k[\pi]]$. By consideration of the lengths of these terms it follows that $j = k$. Consequently $\{\psi_j : \psi_j$ is the associate of θ of type $\eta : j$ and $j \in r\}$ satisfies the subterm condition.

Case II. $C\theta \neq 0$. Note $|C\theta^{[\eta_i]}[\varphi_i]|$ is an increasing function of η_i. For each $i \in r$ pick η_i so large that $|C\theta^{[\eta_i]}[\varphi_i]| > |C\theta|$. Now suppose γ is a non-variable subterm of θ and ☆ holds for some terms π and α. Then (2) must hold and consequently $|C\theta^{[\eta_i]}[\varphi_i]| + |V\theta^{[\eta_i]}[\varphi_i]| \cdot |C\pi| = |C\pi| = |C\xi_i| + |V\xi_i| \cdot |C\alpha|$. Therefore $|V\xi_i| \neq 0$ for each $i \in r$. So by algebraic manipulations

$$|C\xi_1| - \frac{|V\xi_0|}{|V\xi_1|} \cdot |C\xi_0| = \left[|C\theta^{[\eta_1]}[\varphi_1]| - \frac{|V\xi_1|}{|V\xi_0|} \cdot |C\theta^{[\eta_0]}[\varphi_0]|\right]$$

$$+ \left[|V\theta^{[\eta_1]}[\varphi_1]| - \frac{|V\xi_1|}{|V\xi_0|} \cdot |V\theta^{[\eta_0]}[\varphi_0]|\right] \cdot |C\pi| .$$

As γ ranges through the appropriate subterms of θ the left-hand side of this equation takes on only finitely many values. Without loss of generality, assume $|V\varphi_1| \neq 0$. Pick η_1 so large that

$$|C\theta^{[\eta_1]}[\varphi_1]| > \frac{|V\xi_1|}{|V\xi_0|} \cdot |C\theta^{[\eta_0+1]}[\varphi_0]| + \left(|C\xi_1| - \frac{|V\xi_1|}{|V\xi_0|}|C\xi_0|\right) .$$

and

$$|V\theta^{[n_1]}[\varphi_1]| > \frac{|V\xi_1|}{|V\xi_0|} \cdot |V\theta^{[n_0+1]}[\varphi_0]|$$

for each appropriate γ, non-variable subterm of θ. By arguing as in case I from this point on the proof is complete.

Theorem 2.30. *Let $\theta = Q\varphi_0 \dots \varphi_{r-1}$, where Q is an operation symbol of rank $r > 1$ and $\varphi_0, \dots, \varphi_{r-1}$ are terms and $V\theta \neq 0$. There is a set Σ of terms in the same operation symbols as θ such that*

(i) *Σ is infinite,*

(ii) *Σ satisfies the subterm condition,*

(iii) *If $\Delta \cup m(\theta)$ absorbs θ for Γ then $\Delta \cup \Sigma$ for Γ and $\Gamma\{\psi \approx \theta[v_0, v_0, v_0, \dots]: \psi \in \Sigma\}$.*

Proof. Let $\theta' = \theta[v_0, v_0, v_0, \dots]$. Each term that absorbs θ for Γ absorbs θ' for Γ. By Lemma 2.29 there is a set Σ' of associates of θ' such that Σ' has at least two elements and Σ' satisfies the subterm condition. All associates of θ' are absorbed by all sets Δ, for any Γ such that $\Delta \cup m(\theta)$ absorbs θ for Γ. By Theorem 2.23 and Example 2.8 there is an infinite set, Σ, of terms obtained from Σ' by an interpretation operator in such a way that Σ has all the desired properties.

It would be desirable to extend Theorem 2.30 to any non-trivial term θ in which a variable occurs. Unfortunately, this turns out to be impossible as shown in the next example. This example also serves to demonstrate a set of terms which is jointly ω universal but fails to satisfy the subterm condition.

Example 2.31. Let f be a unary operation symbol and B be a binary operation symbol and $\theta = fBv_0fv_0$. For every term, φ, such that $\theta \approx v_0 \vdash \varphi \approx v_0$, the set $\{\varphi\}$ fails to satisfy the subterm condition. $\{\theta\}$ is jointly universal in every cardinal.

Proof. To see that $\{fBv_0fv_0\}$ is jointly κ universal let $g: \kappa \to \kappa$ and note that $\langle \kappa, g, pj_0 \restriction \kappa \rangle$, where pj_0 is the 0th projection function, insures joint κ universality.

Let $\mathfrak{A} = \langle \omega, \bar{f}, \bar{B} \rangle$ where

$$\bar{f}n = \begin{cases} n-1 & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{B}nm = n+1 \quad \text{for all} \quad n,m \in \omega.$$

Let $\mathfrak{B} = \langle Z, \hat{f}, \hat{B} \rangle$ where Z is the set of integers and

$$\hat{f}n = -n \quad \text{for all} \quad n \in Z,$$

$$\hat{B}nm = \begin{cases} m & \text{if } m = -n \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $\mathfrak{A}$ and $\mathfrak{B}$ are models of $\theta \approx v_0$ and therefore they are both models of $\varphi \approx v_0$. Since 0 is not in the range of $\bar{B}$ but $\varphi^{\mathfrak{A}}$ is onto ω, it follows that φ cannot begin with B and so must begin with f (unless $\varphi = v_0$ in which case $\{\varphi\}$ cannot satisfy the subterm condition by definition). Notice that $\{0\}$ is the range of $(Bv_0v_0)^{\mathfrak{B}}$ and $\hat{B}n0 = \hat{B}0n = 0$ for $n \in Z$. Hence fv_0 must be a subterm of φ. Since $\theta \approx v_0 \nvdash fv_0 \approx v_0$, it follows that $\{\varphi\}$ cannot satisfy the subterm condition.

Example 2.31 indicates only some condition weaker than the subterm condition is useful and even necessary for the proofs of the main theorems of this paper. In order to get the most information from the next theorem consider the following definition.

Definition 2.32. For any term θ, θ^+ is defined by recursion:

(i) v_i^+ is v_i, for $i \in \omega$.

(ii) $(f\varphi)^+$ is φ^+, for any term φ and unary operation symbol, f.

(iii) $(Q\varphi_0 \ldots \varphi_{r-1})^+$ is $Q\varphi_0^+ \ldots \varphi_{r-1}^+$, for any operation symbol, Q, of rank $r \neq 1$ and any terms $\varphi_0, \ldots, \varphi_{r-1}$.

So θ^+ is the term obtained from θ by deleting all unary operation symbols. Whenever Σ is a set of terms $\Sigma^+ = \{\theta^+ : \theta \in \Sigma\}$.

Theorem 2.33. *Let θ be any non-trivial term in which a variable occurs. There is a set Σ of terms in the operation symbols of θ such that*

(i) *Σ is infinite,*

(ii) *Σ is jointly κ universal for every infinite cardinal κ,*

(iii) *For any set Γ of equations and any set Δ of terms if $\Delta \cup m(\theta)$ absorbs θ for Γ then $\Delta \cup \Sigma$ absorbs Σ for Γ and $\Gamma \vdash \{\psi \approx \varphi : \psi, \varphi \in \Sigma\}$,*

(iv) *If an operation symbol of rank different from one occurs in θ then Σ^+ satisfies the subterm condition.*

Proof. By virtue of Theorem 2.26 and Corollary 2.6 this theorem is established unless an operation symbol of rank at least two occurs in θ. So suppose $\theta = HQ\varphi_0 \ldots \varphi_{r-1}$, where H is a string (possibly empty) of unary operation symbols, Q is an operation symbol of rank $r > 1$, and $\varphi_0, \ldots, \varphi_{r-1}$ are terms. Without any loss of generality assume v_0 is the only variable occurring in θ. By Lemma 2.29 there is $\eta \in {}^r\omega$ such that $\{\psi_j : j \in r$ and ψ_j is the associate of θ^+ of type $\eta : j\}$ satisfies the subterm condition. For $j \in r$, let

$$\psi_j^* = HQ\theta^{[\eta_0]}[\varphi_0] \ldots \theta^{[\eta_j+1]}[\varphi_j] \ldots \theta^{[\eta_{r-1}]}[\varphi_{r-1}] .$$

Claim. $\{\psi_j^* : j \in r\}$ is jointly κ universal for every infinite cardinal κ.

Proof. Observe that ψ_j^{*+} is ψ_j – the associate of θ^+ of type $\eta : j$, for each $j \in r$. Since $\{\psi_j : j \in r$ and ψ_j is the associate of θ^+ of type $\eta : j\}$ is jointly κ universal by Corollary 2.6 the task is simple. Any assignment of functions over κ to $\{\psi_j^* : j \in r\}$ which agrees according to rank is already an assignment to $\{\psi_j : j \in r$ and ψ_j is the associate of θ^+ of type $\eta : j\}$ that agrees according to rank. Any algebra realizing the assignment for the latter set can be expanded to an algebra realizing the assignment for the former set by setting all unary operations to the identity.

Let

$$\Sigma = \{\psi_0^{*[2]}[\psi_1^{*[n+1]}[\psi_0^*[\psi_1^*]]] : n \in \omega\} .$$

By the claim, Theorem 2.19 and Example 2.8 Σ is jointly κ universal for every infinite cardinal, κ. Clearly Σ is infinite and furthermore, Σ^+ is just the set shown to satisfy the subterm condition in Theorem 2.30. Now suppose $\Delta \cup m(\theta)$ absorbs θ for Γ. Recall $m(\theta) = \{\theta, \varphi_0, \ldots, \varphi_{r-1}\}$. Hence

$$\Gamma \vdash \{\psi_0^* \approx \theta,\ \psi_1^* \approx \theta,\ \psi_0^*[\psi_1^*] \approx \theta,\ \psi_1^*[\psi_0^*] \approx \theta,$$
$$\psi_0^*[\psi_0^*] \approx \theta,\ \psi_1^*[\psi_1^*] \approx \theta\} .$$

Consequently $\Gamma \vdash \{\pi \approx \theta ; \pi \in \Sigma\} \cup \{\pi[\gamma] \approx \theta : \pi, \gamma \in \Sigma\}$ and so $\Delta \cup \Sigma$ absorbs Σ for Γ. The proof is complete.

Theorem 2.34. *Let θ be any term in which at least two distinct variables occur. There is a set Σ of terms in the operation symbols of θ such that*

(i) $|\Sigma \cap \{\varphi : \varphi$ *is a term and* $\mathrm{V}\varphi = n + 1\}| = \omega$ *for every* $n \in \omega$,

(ii) Σ *is jointly* κ *universal for every infinite cardinal* κ,

(iii) *For any set* Γ *of equations and any set* Δ *of terms if* $\Delta \cup m(\theta)$ *absorbs* θ *for* Γ *then* $\Delta \cup \Sigma$ *absorbs* Σ *for* Γ *and* $\Gamma\{\psi \approx \varphi : \varphi, \psi \in \Sigma\}$,

(iv) Σ^+ *satisfies the subterm condition.*

Theorem 2.34 is an extended version of Theorem 2.33 which will not be used in this paper. It is stated here only for completeness. Theorem 2.34 can be obtained from 2.33 by means of a relatively simple construction, the existence theorem and the reduction theorem. The details are omitted in the interest of brevity.

Remarks on the history of reduction

This section has been devoted to notions connected with reduction. The reduction method has been used frequently in the past. Also many arguments are known from the literature which do not require the method in its full extension but in which various notions like the subterm condition play a decisive role. It seems that the underlying idea has occurred to many people. The crux of the concept is relfected in "$F^*\Sigma \vdash F\epsilon$ only if $\Sigma \vdash \epsilon$" as explained following Remark 2.1. This notion admits the possibility of application in many systems or formalisms that embody ideas like consequence or production. It is therefore not surprising that the reduction method finds its sources in sentential logic on the one hand and on the other, in the various combinatorial systems, like Post normal systems and Thue systems, that were introduced in the study of algorithms.

In sentential logic arguments using the crucial idea behind reduction can be traced to the decade 1920–30. I have in mind Theorems 11, 12, 13, and 28 in [15]. For example, Theorem 28 asserts that there are 2^ω complete systems of sentential logic. In sentential logic there are formulas but no terms. Consequently, there can be no subterm condition. But Tarski proved these theorems with help of a "subformula condition" which shares the following property with the subterm condition:

> If Δ satisfies the subformula condition and φ is consequence of Δ, then φ is a substitution instance of a formula in Δ.

At a later date the subterm condition was tacitly introduced into equational logic to obtain a result entirely analogous to Theorem 28 of [15]. [13] establishes the existence of 2^ω equationally complete theories in one binary operation and in the process constructs an infinite set of terms which happens to satisfy the subterm condition. More recently the subterm condition is implicit in [5], [11], [12] and [2], where various extensions of Kalicki's result are proved.

In combinatorial systems related to the theory of algorithms the idea appears in [31] (see p. 268 where Post refers to an earlier paper). In [31] it is established that the correspondence problem for Post normal systems on three letters is not recursively solvable. Post then observes that this result is also true for Post normal systems on two letters and shows how this reduction works by offering appropriate definitions of three letters in terms of two letters and proceeding much as in the present paper. Later in [31], where he shows that the word problem for a specified Thue system on many letters is not recursively solvable, Post remarks that the same reduction techniques apply. This application was actually carried out in [8] where it is essentially shown that even over semigroups countably many constants may be reduced to two constants. This contrasts with Theorem 2.14 and is probably the historical source of Example 2.9. Hall proves a case of the reduction theorem limited to equations without variables. In [17] Hall's result is translated into a similarity type with two unary operations. Also about 1966 Tarski used the reduction method to obtain an essentially undecidable equational theory in one binary operation. [Cf. Theorem 3.6 below.] [29] includes in the proof of one of its last theorems a version of Theorem 2.9 and with it what amounts to another special case of the reduction theorem. V.L. Murskii [26] sketches a proof of a part of Theorem 3.12 below with the help of a condition slightly stronger than the subterm condition. D. Pigozzi has been able to show that if δ is a one-to-one system of definitions for σ in τ such that σ and τ are recursive and the range of δ satisfies the subterm condition and is recursive then $\Theta[\Sigma]_\sigma$ has the same Turing degree as $\Theta[\mathrm{in}_\delta^*\Sigma]_\tau$, for every $\Sigma \subseteq \mathrm{Eq}_\sigma$. Finally, it should be noted that Corollary 2.6 is a very natural extension of a theorem of Isbell [10] which concerns only single terms in unary operation symbols.

Open problems raised by Section 2

(1) [After Jan Mycielski] Let σ be a recursive similarity type. Is $\{\Gamma: |\Gamma| < \omega$ and $\Gamma \subseteq \mathrm{Te}_\sigma$ and Γ is jointly ω universal$\}$ a recursive set? It is not difficult to show that $\{\Gamma: |\Gamma| < \omega$ and $\Gamma \subseteq \mathrm{Te}_\sigma$ and Γ satisfies the subterm condition$\}$ is recursive.

(2) Let κ be an infinite cardinal and Δ be a set of terms such that every finite subset of Δ is jointly κ universal. Is Δ necessarily jointly κ universal?

(3) Can the following converse to the reduction theorem be established? "Let σ and τ be similarity types and let δ be a system of definitions for σ in τ. If in_δ is a reduction of σ to τ then the range of δ is jointly $|\sigma| + \omega$ universal".

(4) Let κ and λ be cardinals, let σ be a similarity type and let $\kappa \to_\sigma \lambda$ denote that every jointly κ universal set of terms of type σ is jointly λ universal. Describe $\to_\sigma$ as a relation between cardinals for different similarity types, e.g. $\sigma = \{\langle Q, 2\rangle\}$.

(5) Let Σ be a set of terms. Define $\Box\Sigma = \{n: n \in \omega$ and Σ is jointly n universal$\}$. For what sets $S \subseteq \omega$ is there a set Σ of terms so that $\Box\Sigma = S$?

3. Base undecidable equational theories

A finitely based equational theory T is base undecidable if the collection of finite bases of T is not recursive. This notion has a sound intuitive base if the similarity type of T is recursive. This section is devoted to the presentation of a quite general condition sufficient to insure that most common finitely based equational theories are base undecidable. Evidently any finitely based equational theory that is undecidable must also be base undecidable. In his doctoral thesis, P. Perkins showed that the equational theory of a one-element groupoid is base undecidable. This result may be found in [29]. In [36] the question is raised as to which finite algebras turn out to be base undecidable. In particular, Tarski suggests there that the equational theory of Boolean algebra may be base undecidable. As consequences of theorems in this section many equational theories of finite algebras turn out to be base undecidable, including the equational theory of Boolean algebras. Some base decidable equational theories of finite algebras will be presented in section 4. A theorem announced in [26] is only slightly weaker than Theorem 3.12(ii) below. The results presented in this paper and those announced by V.L. Murskii were obtained independently and essentially simultaneously.

Definition 3.0.

(i) T is a *base decidable* equational theory if and only if T is an equational theory in a recursive similarity type and $\{\Gamma: |\Gamma| < \omega$ and Γ is a base for $T\}$ is recursive.

(ii) T is a *base undecidable* equational theory if and only if T is a finitely based equational theory and T is not base decidable.

(iii) T is an *essentially base undecidable* equational theory if and only if T is base undecidable and every finitely based theory extending T (perhaps in a similarity type differing from that of T) is base undecidable.

Definition 3.1.

(i) T is a *decidable* equational theory if and only if T is an equational theory and T is recursive.

(ii) T is an *undecidable* equational theory if and only if T is an equational theory and T is not recursive.

(iii) T is an *essentially undecidable* equational theory if and only if T is an undecidable equational theory and every consistent extension of T (perhaps in a similarity type differing from that of T) is also undecidable.

Theorem 3.2. *If T is a finitely based undecidable equational theory then T is base undecidable.*

Proof. Let Γ be any finite base for T and notice that for any equation, ϵ, in the similarity type of T, $\Gamma \cup \{\epsilon\}$ is a base for T if and only if $\epsilon \in T$. Consequently, decision procedure for $\{\Gamma: |\Gamma| < \omega$ and Γ is a base for $T\}$ would yield a procedure for T.

The well-known word problems for Thue systems provide early examples of finitely based undecidable equational theories. In this connection, Post [31] presents a Thue system whose word problem has a negative solution. A Thue system on n letters can be construed as equational by interpreting the letters as constant symbols, juxtaposition as a binary operation symbol, the Thue equivalence symbol as the equality symbol, and by including the associative law in the set of productions of the Thue system under this interpretation. The equations without variables derivable from a Thue system construed in this way will coincide with the productions of the Thue system.

Tarski announced that a certain finitely based equational theory connected with relation algebras is essentially undecidable in [34]. A finitely based undecidable equational theory in two unary operation symbols, as well as various finitely based undecidable theories of loops and quasigroups, was presented in [17]. The work of Mal'cev proves most useful for this paper.

Theorem 3.3. (Mal'cev [17]) *There is a finitely based equational theory T in two unary operation symbols such that*

(i) *If $\varphi \approx \psi \in T$ then $V\varphi = V\psi$ and φ is a variable just in case ψ is a variable,*

(ii) *T is undecidable.*

Mal'cev's theorem is not difficult to prove on the basis of the version of the Post–Markov result about the unsolvability of the word problem for some finitely presented semigroup on two generators described by M. Hall [8]. Since every semigroup is isomorphic to a semigroup of functions over some set Mal'cev's theorem follows. Throughout this section f and g are used to denote the two unary operations involved here and M is a fixed, though otherwise arbitrary finite irredundant base of T.

The next theorem is a restatement of a theorem due independently to McKenzie and to Tarski and announced in [36].

Theorem 3.4. (cf. Tarski [36]). *There is a finite consistent set Δ of equations in a recursive similarity type and there is a recursive function F whose range is included in the class of all equations such that for any finite set Σ of equations in a recursive similarity type $\Delta \cup \Sigma \vdash F\Sigma$ and $F\Sigma \vdash \Sigma \cup \Delta$.*

In fact, Δ turns out to be a certain set of equations closely related to the equational theory of rings. The next theorem seems to be well known but I have been unable to find it mentioned in the literature. Here it may be easily established.

Theorem 3.5. *Let σ be any non-trivial similarity type. There is a finitely based undecidable equational theory of similarity type σ.*

Proof. Combine Theorem 3.3 with the existence theorem and the reduction theorem.

This result can be sharpened if σ has an operation of rank more than one.

Theorem 3.6.[2] *Let σ be a similarity type with an operation symbol of rank more than one. There is a one-based equational theory of type σ which is essentially undecidable. Moreover, if σ has a binary operation symbol then there is a one-based equational theory of type σ which includes the commutative law and is essentially undecidable.*

The major effort of this section is devoted to decidable equational theories that are, however, base undecidable. Because of the reduction

[2] As remarked at the end of Section 2, the first part of this theorem is due to A. Tarski. Tarski found this part in 1966.

theorem it is profitable to first examine theories of a very convenient similarity type σ_0 whose only operation symbols are f, g, h, and k and each of these is unary. σ_0 is to be a recursive similarity type. M, the fixed base for Mal'cev's theory satisfying Theorem 3.3, is a set of equations of type σ_0. First consider whether $\{\Gamma: |\Gamma| < \omega$ and $\Gamma \subseteq \mathrm{Eq}_{\sigma_0}$ and Γ is a base for $\mathrm{Eq}_{\sigma_0}\}$ is recursive – i.e. whether Eq_{σ_0} could be base decidable. In order to see that Eq_{σ_0} is base undecidable consider the following definition of a potential base for Eq_{σ_0}.

Definition 3.7.

$$B(\varphi \approx \psi) = M \cup \{h\varphi[kv_0] \approx h\varphi[kv_1], h\psi[kv_0] = v_0]\}$$

for every equation $\varphi \approx \psi$ in the operation symbols f and g.

In order to show that $\{B(\varphi \approx \psi): \varphi, \psi$ are terms in f and g and $B(\varphi \approx \psi)$ is a base for $\mathrm{Eq}_{\sigma_0}\}$ is not recursive the following lemma is useful.

Lemma 3.8. *Let σ be a similarity type. For any algebra $\mathfrak{A}$ of similarity type σ there is an algebra $\mathfrak{B}$ of type σ such that*

(i) $|B| = |A| + |\sigma| + \omega$;

(ii) *If* $\varphi, \psi \in \mathrm{Te}_\sigma$ *and* $\mathrm{V}\varphi = \mathrm{V}\psi$ *and* $\mathfrak{A} \models \varphi \approx \psi$ *then* $\mathfrak{B} \models \varphi \approx \psi$;

(iii) *If* $\varphi, \psi \in \mathrm{Te}_\sigma$ *and* $\mathrm{V}\varphi = \mathrm{V}\psi$ *and* $\mathfrak{A} \not\models \varphi \approx \psi$ *then there are* $a \in {}^{|B|}({}^{\mathrm{V}\varphi}B)$ *and* $b, c \in {}^{|B|}B$ *such that a, b, and c are one-to-one, the range of b is disjoint from the range of c, and for each* $i \in |B|$, $\varphi^{\mathfrak{B}^*} a_i = b_i$ and $\psi^{\mathfrak{B}^*} a_i = c_i$.

Proof. Let $\infty \notin A$ and define $\mathfrak{A}'$ so that $A' = A \cup \{\infty\}$ and for each Q in the domain of σ and $\alpha \in {}^{\sigma Q}A$ let $Q^{\mathfrak{A}'}\alpha = Q^{\mathfrak{A}}\alpha$ and for $\alpha \in {}^{\sigma Q}A' \sim {}^{\sigma Q}A$ let $Q^{\mathfrak{A}'}\alpha = \infty$. Let $\kappa = |A| + |\sigma| + \omega$ and finally let $\mathfrak{B}$ be the subalgebra ${}^{\kappa}\mathfrak{A}'$ generated by (and even with universe) $B = \{\alpha: \alpha \in {}^{\kappa}A'$ and at most one element of the range of α is different from $\infty\}$. So $|B| = \kappa$ and $\mathfrak{B}$ satisfies conclusion (ii) of the theorem. Now suppose $\varphi, \psi \in \mathrm{Te}_\sigma$, $\mathrm{V}\varphi = \mathrm{V}\psi$, and $\mathfrak{A} \not\models \varphi \approx \psi$. There are $d \in {}^{\mathrm{V}\varphi}A$ and $e, e' \in A$ such that $\varphi^{\mathfrak{A}^*} d = e$ and $\psi^{\mathfrak{A}^*} d = e'$ and $e \neq e'$. Let $a \in {}^{\kappa}({}^{\mathrm{V}\varphi}B)$ such that $a_{ij} = \langle\infty, \infty, ..., d_j, \infty, ...\rangle$ where d_j occurs at the ith place, for all $i \in \kappa$ and $j \in \mathrm{V}\varphi$. Similarly $b, e \in {}^{\kappa}B$ are defined so that $b_i = \langle\infty, \infty, ..., e, \infty, ...\rangle$ and $c_i = \langle\infty, \infty, ..., e', \infty, ...\rangle$ where e and e' occur at the ith place, for each $i \in \kappa$. Then $\varphi^{\mathfrak{B}^*} a_i = b_i$ and $\psi^{\mathfrak{B}^*} a_i = c_i$ for all $i \in \kappa$ and (iii) holds.

Theorem 3.9.

(i) *For all* $\varphi, \psi \in \mathrm{Te}_{\sigma_\omega}$ *if* $M \vdash \varphi \approx \psi$ *then* $B(\varphi \approx \psi) \vdash \varphi \approx \psi$.

(ii) *For all terms* φ, φ', ψ *and* ψ' *in the operation symbols* f *and* g *such that* $V\varphi = V\psi$ *and* $V\varphi' = V\psi'$ *if* $M \nvdash \varphi \approx \psi$ *then* $B(\varphi \approx \psi) \vdash \varphi' \approx \psi'$ *just in case* $M \vdash \varphi' \approx \psi'$.

(iii) Eq_{σ_0} *is essentially base undecidable.*

Proof. (i) is immediate since $M \subseteq B(\varphi \approx \psi)$. (iii) follows easily from (ii) by means of Theorem 3.3. So suppose φ and ψ are terms in f and g such that $V\varphi = V\psi$ and $M \nvdash \varphi \approx \psi$. Then there is an algebra $\mathfrak{A}$ of cardinality ω such that $\Theta[M] = \mathrm{Th}\,\mathfrak{A}$. By the Lemma there is an algebra $\mathfrak{B}$ of cardinality ω such that $\mathfrak{B} \models M$ and there are $a, b, c \in {}^\omega B$ such that a, b, and c are one-to-one, the range of b is disjoint from the range of c and $\varphi^{\mathfrak{B}^*} a_i = b_i$ and $\psi^{\mathfrak{B}^*} a_i = c_i$ for every $i \in \omega$. Consequently $\mathfrak{B} \not\models \varphi \approx \psi$. $\mathfrak{B}$ is to be expanded to a model of $B(\varphi \approx \psi)$. To this end let $\bar{k}$ be a one-to-one map from B onto the range of a and let $\bar{h}$ be defined in any way such that $\bar{h} b_i = a_0$ and $\bar{h} c_i = \bar{k}^{-1} a_i$, for every $i \in \omega$. Then $(\mathfrak{B}, \bar{h}, \bar{k})$ is a model of $B(\varphi \approx \psi)$ and the theorem is established.

Of course the base undecidability of Eq_{σ_0} seems a very special result. Nevertheless, it is the foundation that will be used to establish the base undecidability of a much wider class of theories. It is first necessary to extend the definition of $B(\varphi \approx \psi)$.

Definition 3.10. Let σ be a similarity type, δ be a system of definitions for σ_0 in σ, $\Gamma \subseteq \mathrm{Eq}_\sigma$ and φ, ψ be terms in f and g. $B(\varphi \approx \psi, \delta, \Gamma)$ is the set $\mathrm{in}_\delta^* M \cup \{(\mathrm{in}_\delta h\varphi[kv_0])[\gamma_l] \approx (\mathrm{in}_\delta h\varphi[kv_1])[\gamma_r]: \gamma \in \Gamma\} \cup \{(\mathrm{in}_\delta h\psi[kv_0])[\gamma] \approx \gamma: \gamma \in t\Gamma\}$.

Theorem 3.11. *Let* σ *be a similarity type,* δ *a system of definitions for* σ_0 *in* σ *such that* in_δ *is a reduction. Let* $\Gamma \subseteq \mathrm{Eq}_\sigma$ *such that for any* φ, ψ *in the range of* δ, $t\Gamma$ *together with the range of* δ *absorbs* φ *for* Γ *and* $\Gamma \vdash \varphi \approx \psi$. *Let* γ *and* ϵ *be equations in* f *and* g *such that* $V\gamma_r = V\gamma_l$ *and* $V\epsilon_r = V\epsilon_l$ *and* ϵ_r *is a variable just in case* ϵ_l *is also, then*

(i) $M \vdash \epsilon$ *if and only if* $\Theta[B(\epsilon, \delta, \Gamma)]_\sigma = \Theta[\Gamma]_\sigma$;

(ii) *If* $M \nvdash \epsilon$ *then* $M \vdash \gamma$ *just in case* $B(\epsilon, \delta, \Gamma) \vdash \mathrm{in}_\delta \gamma$.

Proof. Since in_δ is a reduction $M \vdash \epsilon$ if and only if $\mathrm{in}_\delta^* M \vdash \mathrm{in}_\delta \epsilon$ and $B(\epsilon) \vdash \gamma$ if and only if $\mathrm{in}_\delta^* B(\epsilon) \vdash \mathrm{in}_\delta \gamma$. Now $\Gamma \vdash B(\epsilon, \delta, \Gamma) \cup \{\mathrm{in}_\delta \epsilon\}$ by the absorption hypothesis of the theorem and the properties of M described by Theorem 3.3. Furthermore, $\text{in}_\delta^* B(\epsilon) \vdash B(\epsilon, \delta, \Gamma)$ since each member of $B(\epsilon, \delta, \Gamma)$ is a substitution instance of $\text{in}_\delta^* B(\epsilon)$. Obviously $B(\epsilon, \delta, \Gamma) \cup \{\text{in}_\delta\, \epsilon\} \vdash \Gamma$, so $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \epsilon$ if and only if $\Theta[B(\epsilon, \delta, \Gamma)]_\sigma = \Theta[\Gamma]_\sigma$. Moreover, if $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \epsilon$ then $\text{in}_\delta^* B(\epsilon) \vdash \text{in}_\delta\, \epsilon$ and so $B(\epsilon) \vdash \epsilon$, since in_δ is a reduction. Consequently, if $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \epsilon$ then $M \vdash \epsilon$ by Theorem 3.9. Conversely, if $M \vdash \epsilon$ then $\text{in}_\delta^* M \vdash \text{in}_\delta\, \epsilon$ and so $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \epsilon$. Therefore $M \vdash \epsilon$ if and only if $\Theta[\Gamma]_\sigma = \Theta[B(\epsilon, \delta, \Gamma)]_\sigma$ and (i) is established. Now suppose $M \nvdash \epsilon$, then if $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \gamma$ then $\text{in}_\delta^* B(\epsilon) \vdash \text{in}_\delta\, \gamma$ and hence $B(\epsilon) \vdash \gamma$. By Theorem 3.9, if $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \gamma$ then $M \vdash \gamma$. On the other hand if $M \vdash \gamma$ then $\text{in}_\delta^* M \vdash \text{in}_\delta\, \gamma$ and hence $B(\epsilon, \delta, \Gamma) \vdash \text{in}_\delta\, \gamma$. In this way (ii) is demonstrated and the theorem is proved.

The next theorem can be regarded as the main result of this paper. It provides general conditions sufficient to establish that many familiar finitely based equational theories are base undecidable.

Theorem 3.12. (The base undecidability theorem). *Let σ be a similarity type.*

(i) *If $\Gamma \subseteq \text{Eq}_\sigma$, $|\Gamma| < \omega$, and there is a non-trivial term θ such that $\theta \in \text{Te}_\sigma$, $V\theta \neq 0$, and $t\Gamma \cup m(\theta)$ absorbs θ for Γ then $\Theta[\Gamma]_\sigma$ is base undecidable.*

(ii) *If T is a finitely based equational theory and there is a non-trivial term θ such that $\theta \approx v_0 \in T$ then T is essentially base undecidable.*

Proof. (ii) is an immediate corollary of (i). By Theorem 2.33 there is a system δ of definitions for σ_0 in σ such that δ is one-to-one, the range of δ is jointly ω universal, $t\Gamma$ together with the range of δ absorbs the range of δ for Γ and $\Gamma \vdash \varphi \approx \psi$ for any φ, ψ in the range of δ. By Theorem 2.3, the reduction theorem, in_δ is a reduction. By Theorem 3.11 $B(\epsilon, \delta, \Gamma)$ is a base for $\Theta[\Gamma]_\sigma$ if and only if $M \vdash \epsilon$, for every equation ϵ in f and g such that $V\epsilon_l = V\epsilon_r$ and ϵ_l is a variable just in case ϵ_r is also. It follows from Theorem 3.3 that $\{B(\epsilon, \delta, \Gamma)\colon B(\epsilon, \delta, \Gamma)$ is a base for $\Theta[\Gamma]_\sigma$ and $V\epsilon_l = V\epsilon_r$ and $\epsilon_l \in \text{Va}$ if and only if $\epsilon_r \in \text{Va}\}$ is not recursive. Consequently, $\Theta[\Gamma]_\sigma$ is base undecidable and the theorem is established.

The next few theorems illustrate the extensive range of applications of Theorem 3.12. A theory θ is said to be a theory of groups (rings, lattices, Boolean algebras, ...) if θ is definitionally equivalent to the equational theory of a group (ring, lattice, Boolean algebra, ...) in one of the standard formulations.

Theorem 3.13. *Let T be a finitely based consistent equational theory. T is essentially base undecidable if any of the following hold:*

(i) *T is a theory of groups;*

(ii) *T is a theory of semilattices;*

(iii) *T is a theory of lattices;*

(iv) *T is a theory of rings;*

(v) *T is a theory of rings with unit;*

(vi) *T is a theory of Boolean algebras;*

(vii) *T is a theory of relation algebras.*

Proof. The proofs of the different cases above vary only in details from one another. The basic idea is to discover, in each, a non-trivial term θ in some standard formulation of the theory so that $\theta \approx v_0$ is true and in any definitionally equivalent theory, θ corresponds to a non-trivial term. In particular, if (i) and (ii) are established in this way then the other cases will follow easily.

I. Take the formulation of group theory in which groups are algebras with two operations: composition and the formation of inverses. Let $\cdot$ and $^{-1}$ be the corresponding operation symbols. Let δ be a definition of $v_0 \cdot v_1$ and let γ be a definition of v_0^{-1}, both in the wider sense, for the similarity type of T. Then $2 \subseteq V\delta$ since T is consistent (otherwise $v_0 \cdot v_0 \approx v_0 \cdot v_1$ would hold in the standard formulation of T). Let θ be the term corresponding by δ and γ to $v_0 \cdot (v_0 \cdot v_0^{-1})$. So $\theta \approx v_0 \in T$ and θ is non-trivial. By Theorem 3.12, T is essentially base undecidable.

II. Take the formulation of semilattices as algebras with the operation meet $\wedge$. Let δ be a definition in the wider sense for $v_0 \wedge v_1$ for the similarity type of T. At least the variables v_0 and v_1 occur in δ since T is consistent (otherwise $v_0 \wedge v_0 \approx v_0 \wedge v_1$ would hold in the standard formulation of T). Let θ be $\delta[v_0, v_0]$. Then $\theta \approx v_0 \in T$ and θ is non-trivial so by Theorem 3.12 T is essentially base undecidable.

The remaining cases now follow easily.

Perhaps it should be remarked that what was actually shown in the proof above was that if T is finitely based and T is definitionally equivalent to a theory T' such that T' is an extension of either a theory of groups or a theory of semilattices then T is essentially base undecidable. This should be contrasted with the following result.

Theorem 3.14. *Every finitely based theory is definitionally equivalent with an essentially base undecidable theory.*

Proof. Let T be finitely based and suppose f and g are two unary operation symbols not occurring in T. Then T is definitionally equivalent with $\Theta[T \cup \{fv_0 \approx v_0, gv_0 \approx v_0\}]$ but by Theorem 3.12 this last theory is essentially base undecidable.

Theorem 3.15. *Let σ be a non-trivial similarity type.* Eq_σ *is essentially base undecidable.*

Proof. Let θ be a non-trivial term in type σ. Eq_σ is finitely based and $\theta \approx v_0 \in \mathrm{Eq}$ so Theorem 3.12 applies.

In the case that σ has an operation symbol of rank at least two Theorem 3.15 is essentially contained in [29].

Theorem 3.16. *If T is a finitely based essentially undecidable equational theory then T is essentially base undecidable.*

Proof. Since no finitely based theory in a recursive trivial similarity type can be undecidable, the theorem follows from Theorem 3.2 and Theorem 3.15.

Definition 3.17. Let σ be a similarity type. T is the *constant theory* of type σ if and only if T is an equational theory of type σ and for all $\mathfrak{A} \models T$ and all i, j in the domain of σ, $\mathrm{Op}\mathfrak{A}_i$ has the same range as $\mathrm{Op}\mathfrak{A}_j$ and the range of $\mathrm{Op}\mathfrak{A}_i$ has exactly one element.

T is the constant theory of type σ if and only if $T = \{\varphi \approx \psi : \varphi, \psi \in \mathrm{Te}_\sigma \sim \mathrm{Va}\} \cup \{v_i : i \in \omega\}$. For any similarity type σ the constant theory of type σ is equationally complete. For non-trivial finite similarity types the constant theories are base undecidable.

Theorem 3.18. *Let σ be a finite non-trivial similarity type. The constant theory of type σ is base undecidable.*

Proof. Let θ be any non-trivial term of similarity type σ. θ is absorbed by every subset of $\mathrm{Te}_\sigma \sim \mathrm{Va}$ for T. Since σ is finite T must be finitely based.

Let Γ be any finite base for T such that $t\Gamma \cap Va = 0$. By Theorem 3.12 T is base undecidable.

Theorem 3.19. *Let σ be a similarity type with some operation symbol of rank at least two or at least three unary operation symbols. Every consistent finitely based equational theory of type σ has a consistent base undecidable extension of type σ.*

Proof. Let $\Gamma \subseteq \mathrm{Eq}_\sigma$ and $|\Gamma| < \omega$. For the moment assume that there is a non-variable term θ such that $\Gamma \vdash \theta \approx v_0$. If θ is non-trivial then $\Theta[\Gamma]_\sigma$ is base undecidable. Suppose θ is trivial. There is a unary operation symbol, say f, and a natural number $n > 0$ so that $\theta = f^n v_0$. If $m \in \omega$ and $\pi \in \mathrm{Te}_\sigma$ such that $\Gamma \vdash f^m v_0 \approx \pi$ then $\Gamma \vdash f^{m\cdot(n-1)}\pi \approx f^{m\cdot n} v_0$ so $\Gamma \vdash f^{m\cdot(n-1)}\pi \approx v_0$. So either $\Theta[\Gamma]_\sigma$ is base undecidable or else no operation symbol different from f occurs in π whenever $\Gamma \vdash \pi \approx f^m v_0$. Assume that $\Theta[\Gamma]_\sigma$ is not itself base undecidable. Let σ' be a similarity type which is the restriction of σ to a finite set of operation symbols different from f which include all the operation symbols different from f occurring in Γ and such that σ' is non-trivial. Let T' be the constant theory of type σ'. T' is base undecidable by Theorem 3.18. Let $T = \Theta[T' \cup \{fv_0 \approx v_0\}]_\sigma$. Evidently, $T \supseteq \Theta[\Gamma]_\sigma$, T is finitely based, and every model of T' can be expanded to a model of T. Hence T is consistent. Let $\Delta \subseteq \mathrm{Eq}_\sigma$. Δ is a base for T' if and only if $\Delta \cup \{fv_0 \approx v_0\}$ is a base for T. (Suppose $\Delta \cup \{fv_0 \approx v_0\}$ is a base for T. Let $\mathfrak{A} \models T'$ and expand $\mathfrak{A}$ to $\mathfrak{B} \models T$. Then $\mathfrak{B} \models \Delta$ and hence $\mathfrak{A} \models \Delta$. This means $T' \vdash \Delta$. Let $\mathfrak{A} \models \Delta$ and expand $\mathfrak{A}$ to $\mathfrak{B} \models T$. Then $\mathfrak{B} \models T'$ and hence $\mathfrak{A} \models T'$. This means $\Delta - T'$.) Since T' is base undecidable it follows that T is base undecidable. Now suppose there is no non-variable term θ such that $\Gamma \vdash \theta \approx v_0$. In this case let σ' be the restriction of σ to any finite non-trivial set of operation symbols including all those occurring in Γ. Let T' be the constant theory for type σ'. Let $T = \Theta[T']_\sigma$. Again T is base undecidable and the proof is complete.

Corollary 3.20. *Let σ be a similarity type with some operation symbol of rank at least two or at least three unary operation symbols. Every finitely based equationally complete theory of type σ is base undecidable.*

In Section 4 an example of a finitely based equationally complete base decidable equational theory in two unary operation symbols will be provided.

Theorem 3.21. *Let σ be a similarity type. If $\Gamma \subseteq \mathrm{Eq}_\sigma$ and there is a nontrivial term θ such that $\nabla\theta \neq 0$ and $m(\theta) \cup t\Gamma$ absorbs θ for Γ then there is an infinite family $\mathfrak{J}$ of finitely based undecidable subtheories of $\Theta[\Gamma]_\sigma$. If, in addition Γ is finite and $\Gamma \vdash \theta \approx v_0$, then $\mathfrak{J}$ can be chosen so that every member of $\mathfrak{J}$ has the same finite models as Γ.*

Proof. By Theorem 2.31 there is a set Σ of terms such that

(i) Σ is infinite.

(ii) Σ is jointly universal in every infinite cardinal.

(iii) $t\Gamma \cup \Sigma$ absorbs Σ for Γ and $\Gamma \vdash \{\psi \approx \varphi\colon \psi, \varphi \in \Sigma\}$.

(iv) There are terms ψ_0, ψ_1 such that $\Sigma = \{\psi_0^{[2]}[\psi_1^{[n+1]}[\psi_0[\psi_1]]]\colon n \in \omega\}$.

Let $\delta \in {}^\omega\Sigma$ such that δ is one-to-one. For every $k \in \omega$ let η_k be the system of definitions for the similarity type of M in σ such that fv_0 is defined by δ_{4k+2} and gv_0 is defined by δ_{4k+3}. Now suppose that Γ is finite and that $\Gamma \vdash \theta \approx v_0$. (The other case can easily be seen by simplifying the argument below). Let $\Delta = \{\delta_0[\gamma_l] \approx \delta_0[\gamma_r]\colon \gamma \in \Gamma\} \cup \{\delta_1 \approx v_0\}$. Define T_k for $k \in \omega$ as the theory based on $\Delta \cup \mathrm{in}^*_{\eta_k} M \cup \{\delta_{4k+4} \approx \delta_{4k+5}\}$.

Claim 1. T_k is undecidable for each $k \in \omega$.
Proof. Let $\mathfrak{A}$ be an algebra such that $\mathrm{Th}\,\mathfrak{A} = \Theta[\mathrm{in}^*_{\eta_k} M]_\sigma$ and $|A| = \omega$. Let $\mathfrak{B}$ be an algebra so that $\delta_0^{\mathfrak{B}}$ has a one-element range, $\delta_1^{\mathfrak{B}*}$ is the identity function on A, $\delta_{4k+2}^{\mathfrak{B}} = \delta_{4k+2}^{\mathfrak{A}}$, $\delta_{4k+3}^{\mathfrak{B}} = \delta_{4k+3}^{\mathfrak{A}}$, $\delta_{4k+4}^{\mathfrak{B}} = \delta_{4k+5}^{\mathfrak{B}}$, and otherwise so that $\delta_{4j+4}^{\mathfrak{B}} \neq \delta_{4j+5}^{\mathfrak{B}}$. Then $\mathfrak{B} \models T_k$ and for every equation ϵ in f and g, $\mathfrak{B} \models \mathrm{in}_{\eta_k}\epsilon$ if and only if $\mathfrak{A} \models \mathrm{in}_{\eta_k}\epsilon$ if and only if $M \vdash \epsilon$. Since $\mathrm{in}^*_{\eta_k} M \subseteq T_k$ this means that T_k is undecidable by Theorem 3.3.

Claim 2. $\Gamma \vdash T_k$, for all $k \in \omega$.
Proof. This is a consequence of (iii).

Claim 3. If $\mathfrak{A}$ is a finite model of T_k then $\mathfrak{A} \models \Gamma$, for all $k \in \omega$.
Proof. $\mathfrak{A} \models \delta_1 \approx v_0$ so $\delta_1^{\mathfrak{A}*}$ is one-to-one and onto. By (iv) and the finiteness of $\mathfrak{A}$ both $\psi_0^{\mathfrak{A}*}$ and $\psi_1^{\mathfrak{A}*}$ must be one-to-one and onto. But this means that $\delta_0^{\mathfrak{A}*}$ is one-to-one and onto and hence invertible. Since $\mathfrak{A} \models \{\delta_0[\gamma_l] \approx \delta_0[\gamma_r]\colon \gamma \in \Gamma\}$, this means that $\mathfrak{A} \models \Gamma$.

The claims complete the proof of the last sentence of the theorem. The alteration needed to prove the first sentence is the deletion of Δ from the definition of T_k. Then claims 1 and 2 still hold and so the theorem follows.

It should be remarked that, in any finite similarity type, the set of finite bases of any finitely based equational theory is recursively enumerable. For every recursively enumerable Turing degree e there is a finitely presented semigroup on two generators which has a word problem of degree e. Such semigroups are provided in [4]. It is therefore possible to obtain a finite set, M_e, of equations in f and g such that $\Theta[M_e]$ has Turing degree e. It follows that for all equational theories T in recursive similarity types which fulfill the hypotheses of the base undecidability theorem, the set of finite bases of T has the largest possible recursively enumerable degree: $0'$.

Problems raised by Section 3

(1) Let σ be a finite non-trivial similarity type. Is the constant theory of similarity type σ essentially base undecidable?

(2) If T is a one-based base undecidable theory, is $\{\epsilon\colon \Theta[\epsilon] = T\}$ ever recursive?

(3) Is the set $\{\mathfrak{A}\colon A \in \omega$ and $\mathfrak{A}$ is a finitely based groupoid$\}$ recursive?

(4) Is there a finitely based undecidable theory of groups? Murskii [25] provides a finitely based undecidable theory of semigroups and Mal'cev [17] provides various finitely based undecidable theories of quasigroups and loops. The Boone–Novikov construction (cf. [3], [27]) yields a finitely based undecidable theory of groups with several additional constants.

(5) The behavior of base undecidability with respect to definitional equivalence is largely unexplored. Is there a reasonable condition, independent of Theorem 3.12, on theories T such that if T is base undecidable then so is every theory definitionally equivalent with T?

Problem 3 is perhaps the most challenging. It was raised in [36], though it has received consideration earlier (cf. P. Perkins' doctoral thesis, Berkeley, 1966).

4. Base decidable theories

The purpose of this Section is to investigate base decidable theories. Considering the base undecidability theorem, it is to be expected that base decidable theories occur infrequently amongst the more familiar finitely based theories. Though many of the examples of base decidable theories presented here are rooted in semigroups, several are simply artificial. Part of the energy that generated these examples stemmed from an investigation of the connection between the base decidability of a theory and whether it could have arbitrarily large finite irredundant bases. The reason for suspecting such a connection is contained in the following theorem.

Theorem 4.0. *Let σ be a similarity type. If $\Gamma \subseteq \mathrm{Eq}_\sigma$, Γ is finite, and there is a non-trivial term θ such that $V\theta \neq 0$ and $m(\theta) \cup t\Gamma$ absorbs θ for Γ then $\Theta[\Gamma]_\sigma$ has irredundant bases of arbitrarily large finite cardinality (i.e. $\nabla\, \Theta[\Gamma]_\sigma$ is infinite).*

Proof. By Theorem 2.33 there is a set Σ of terms so that Σ is infinite and jointly $|\sigma| + \omega$ universal and $\Sigma \cup t\Gamma$ absorbs Σ for Γ and $\Gamma \vdash \{\psi \approx \varphi\colon \psi, \varphi \in \Sigma\}$. Let $\delta \in {}^\omega\Sigma$ such that δ is one-to-one.

For $n \in \omega$ let

$$\Delta_n = \{\delta_i[\gamma] \approx \gamma\colon \gamma \in t\Gamma \text{ and } i \in n\}$$
$$\cup\ \{\delta_0[\delta_1[\ldots\delta_{n-1}[\gamma_l]\ldots] \approx \delta_0[\delta_1[\ldots\delta_{n-1}[\gamma_r]\ldots]\colon \gamma \in \Gamma\}.$$

Then $\Delta_n \vdash \Gamma$ for each $n \in \omega$ and, since $t\Gamma$ absorbs the range of δ for Γ ti follows that $\Gamma \vdash \Delta_n$. Hence $\Theta[\Gamma]_\sigma = \Theta[\Delta_n]_\sigma$. For each $n \in \omega$, let Λ_n be an irredundant subset of Δ_n such that $\Theta[\Gamma]_\sigma = \Theta[\Lambda_n]_\sigma$.

Claim For each $n \in \omega \sim 2$ and $i \in n$ there is $\gamma \in t\Gamma$ such that $\delta_i[\gamma] \approx \gamma \in \Lambda_n$.

Proof. Suppose otherwise and, without loss of generality that

$$\Lambda_n \subseteq \Delta_n \sim \{\delta_{n-1}[\gamma] \approx \gamma\colon \gamma \in t\Gamma\}.$$

Since $\{\delta_j\colon j \in n\}$ is jointly $|\sigma| + \omega$ universal let $\mathfrak{A}$ be an algebra of type σ so that $\delta_j^{\mathfrak{A}*}$ is the identity function on A if $j < n-1$ and $\delta_{n-1}^{\mathfrak{A}*}$ is a constant function. Then $\mathfrak{A} \models \Delta_n \sim \{\delta_{n-1}[\gamma] \approx \gamma\colon \gamma \in t\Gamma\}$. So $\mathfrak{A} \models \Lambda_n$ and therefore $\mathfrak{A} \models \Gamma$. But $\mathfrak{A} \not\models \delta_{n-1} \approx \delta_0$ and thus $\Gamma \nvdash \delta_{n-1} \approx \delta_0$ but this is contrary to $\Gamma \vdash \{\psi \approx \varphi\colon \psi, \varphi \in \Sigma\}$. So the claim is finished.

It follows from the claim that $|\Lambda_n| \geqslant n$ for each $n \in \omega \sim 2$. So the theorem is established.

In [36], it was announced that if T is a finitely based equational theory and there is a term θ in which v_0 occurs at least twice, such that $\theta \approx v_0 \in T$ then ∇T is infinite. Theorem 4.0 extends this result. Theorems 3.12, 3.21 and 4.0 have essentially the same hypotheses, though different conclusions.

Below, a finitely based theory T of commutative semigroups is presented such that T is base decidable, ∇T is infinite, and T has infinitely many finitely based undecidable subtheories. Another finitely based theory T of commutative semigroups is provided that is base decidable so that ∇T is finite, T is the theory of a finite algebra, but T has infinitely many essentially different irredundant bases. It is not known whether every finitely based theory T in a recursive similarity type such that ∇T is finite is base decidable.

Because semigroups play such a role in this section, $\cdot$ is introduced as a binary operation symbol and terms are constructed so that $(\varphi \cdot \psi)$ is the term resulting from applying $\cdot$ to φ and ψ. "(" and ")" must be added to the fundamental symbols of equational logic. The similarity type having $\cdot$ as its only operation symbol is taken to be recursive. Whenever φ is a term in $\cdot$ then φ^1 is φ and φ^{n+1} is $\varphi \cdot \varphi^n$ for each $n \in \omega \sim 1$. This should give rise to no confusion with the notation of previous sections in what follows.

Definition 4.1.[3] Let σ be any similarity type.

(i) For each $i \in \omega$ and $\theta \in \mathrm{Te}_\sigma$, let $i(\theta)$ be the number of times v_i occurs in θ.

(ii) For each Q in the domain of σ and $\theta \in \mathrm{Te}_\sigma$, let $Q[\theta]$ be the number of times Q occurs in θ. $\cdot[\theta]$ is the number of times $\cdot$ occurs in θ.

(iii) For $\Gamma \subseteq \mathrm{Eq}_\sigma$, let $\mathrm{R}\Gamma = \{\gamma_l[\theta] \approx \gamma_r[\theta]$: either $\gamma \in \Gamma$ or $\gamma_r \approx \gamma_l \in \Gamma$ and $\theta \in {}^\omega\mathrm{Te}_\sigma$ and $\mathrm{V}\theta_i = \{0\}$ for each $i \in \omega\}$.

(iv) Let $\epsilon \in \mathrm{Eq}_\sigma$. ϵ is *balanced* if and only if $i(\epsilon_l) = i(\epsilon_r)$ and $Q[\epsilon_l] = Q[\epsilon_r]$, for each $i \in \omega$ and each Q in the domain of σ such that $\sigma Q \nless 1$.

As a matter of convenience the greatest common divisor of the empty set is taken to be 0 and $\min 0 = 0$. $n \mid m$ means n divides m for every $m, n \in \omega$.

Lemma 4.2. *Let Γ be a set of equations in the operation symbol $\cdot$. Let* $m = \min(\{\cdot[\theta] + 1$: *there is a term π such that either* $\pi \approx \theta \in \Gamma$ *or* $\theta \approx \pi \in \Gamma$ *and* $\cdot[\theta] \neq \cdot[\pi]\} \cup \{\cdot[\theta] + i(\theta) + 1$: $i \in \omega$ *and there is a term*

[3] An earlier definition of balanced equation appearing in [21] is incorrect in that operation symbols are ignored. I would like to thank Don Pigozzi for pointing out this mistake. Also the condition that t involves a variable must be added to the hypothesis of the theorem announced in [21].

π such that either $\pi \approx \theta \in \Gamma$ or $\theta \approx \pi \in \Gamma$ and $i(\theta) \neq i(\pi)\}$). Let r be the greatest common divisor of $\{i(\gamma_l) - i(\gamma_r)\colon i \in \omega$ and $\gamma \in \Gamma\}$. Then

(i) $$\Theta[\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\} \cup \mathrm{R}\Gamma\} = \Theta[\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0^m \approx v_0^{m+r}\}]$$

(ii) $$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_1)\} \cup \Gamma \vdash v_0^{m'} \approx v_0^{m'+r'}$$

if and only if $m' \geqslant m$ and $r \mid r'$, for every $m', r' \in \omega \sim 1$.

The proof of this lemma is elaborate but uninteresting, essentially some easy number-theoretical manipulations combined with rules of inference for equational logic. For brevity the proof is not given. In any case, some of the principles used in the proof of this lemma arise again in the proof of the next theorem.

Theorem 4.3. *Let $m \in \omega \sim 2$ and $r \in \omega$. Let T be the theory based on*

$$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2),\ v_0 \cdot v_1 \approx v_1 \cdot v_0,\ v_0^m \approx v_0^{m+r}\}.$$

T is base decidable and ∇T is infinite if $r > 0$.

Proof. If $\gamma \in T$ then $\mathrm{V}\gamma_l = \mathrm{V}\gamma_r$.

Claim 1. $\gamma \in T$ if and only if $r \mid i(\gamma_l) - i(\gamma_r)$ and either $i(\gamma_l) = i(\gamma_r)$ or $\min(i(\gamma_l), i(\gamma_r)) \geqslant m$, for each $i \in \omega$.

Proof. Suppose γ satisfies the conditions on the right side. Let $r_i = |i(\gamma_l) - i(\gamma_r)|$ and $m_i = \min(i(\gamma_l), i(\gamma_r))$. By Lemma 4.2, $v_i^{m_i} \approx v_i^{m_i+r_i} \in T$. Since the commutative law, as well as the associative law, is in T then $\gamma \in T$.

Conversely, suppose $\gamma \in T$. Let $k \in \omega$ such that $\mathrm{V}\gamma_l \subseteq k$ and let $\chi = v_0^m \cdot v_1^m \cdot \ldots \cdot v_k^m$. For $i, j \in \omega$ let

$$\theta_{ij} = \begin{cases} v_0 & \text{if } i = j, \\ v_0^r & \text{if } i \neq j, \end{cases}$$

and $\theta_i = \langle \theta_{i,j}\colon j \in \omega\rangle$. Let $r_i = |i(\gamma_l) - i(\gamma_r)|$ for $i \in \omega$. Let $q \in \omega$.

$$\chi[\theta_q] \cdot \gamma_l[\theta_q] \approx \chi[\theta_q] \cdot \gamma_r[\theta_q] \in T,$$

so there are $p_q \in \omega \sim m$ and $d_q \in \omega$ such that

$$v_0^{p_q + d_q r + r_q} \approx v_0^{p_q} \in T$$

or else

$$v_0^{p_q + r_q} \approx v_0^{p_q + d_q r} \in T.$$

Then by Lemma 4.2 $v_0^{p_q} \approx v_0^{p_q + r_q} \in T$ and by Lemma 4.2 again $r \mid r_q$. Now suppose $q(\gamma_l) > q(\gamma_r)$. Since both the commutative law and the associative law are balanced and since $T \vdash \gamma$, by the definition of derivation there must be a term φ such that either φ^m or φ^{m+r} is a subterm of γ_r and furthermore v_q occurs in φ. Therefore $q(\gamma_r) \geqslant m$. The claim is proven.

In particular, T is decidable.

Claim 2. Let Δ be a set of equations. Δ is a base for T if and only if

(1) $T \vdash \Delta$;

(2) The commutative law is in Δ;

(3) One of the sixteen commutations of the associative law is in Δ;

(4) r is the greatest common divisor of $\{i(\gamma_l) - i(\gamma_r) \colon \gamma \in \Delta \text{ and } i \in \omega\}$;

(5) $m = \min(\{\cdot[\theta] + 1 \colon$ there is a term π such that either $\pi \approx \theta \in \Delta$ or $\theta \approx \pi \in \Delta$ and $\cdot[\theta] \neq \cdot[\pi]\} \cup \{\cdot[\theta] + i(\theta) + 1 \colon i \in \omega$ and there is a term π such that either $\theta \approx \pi \in \Delta$ or $\pi \approx \theta \in \Delta$ and $i(\theta) \neq i(\pi)\})$.

Proof. Suppose (1)–(5) hold. Then $T \vdash \Delta$ and both the commutative and associative laws are derivable from Δ. Lemma 4.2 together with (4) and (5) insures that Δ is a base for T.

To demonstrate the converse, suppose Δ is a base for T. (1) is immediate and (2) and (3) follow from the definition of derivation considering $m \geqslant 2$. By Lemma 4.2(i)

$$\Theta[\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\} \cup \mathrm{R}\Delta] =$$

$$= \Theta[\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0^m \approx v_0^{m+r}\}] .$$

By Lemma 4.2(ii), (4) and (5) are fulfilled. The claim is established.

Claims 1 and 2 provide a decision procedure that determines which finite sets of equations are bases for T and so T is base decidable.

Now suppose $r > 0$. For each $n \in \omega \sim 1$ there is $s_n \in {}^n(\omega \sim 1)$ such that s_n is one-to-one and r is the greatest common divisor of the range of s_n but not of any proper non-empty subset of the range of s_n. Let

$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0 \cdot v_1 \approx v_1 \cdot v_0\} \cup \{v_0^m \approx v_0^{m+s_nj}: j \in n\}$. By Lemma 4.2 Δ_n is a base for T and Δ_n is irredundant by construction. So ∇T is infinite and the theorem is established.

Example 4.4. Let T be the theory based on $\{(v_0 \cdot v_0) \cdot (v_0 \cdot v_0) \approx v_0 \cdot v_0\}$ T is base undecidable but T is not essentially base undecidable. In addition, T has infinitely many finitely based undecidable subtheories but T is itself decidable.

Proof. Let θ be $(v_0 \cdot v_0) \cdot (v_0 \cdot v_0)$. Then θ is non-trivial and $m(\theta) = \{\theta, v_0 \cdot v_0\}$. Observe that $m(\theta)$ absorbs θ for $\{(v_0 \cdot v_0) \cdot (v_0 \cdot v_0) \approx v_0 \cdot v_0\}$. So that by Theorems 3.12 and 3.21 the undecidability results mentioned follow. By Theorem 4.3 the extension of T by the addition of the commutative and associative laws is base decidable. Therefore T is not essentially base undecidable. The decidability of T can either be seen by an exhaustive analysis or by employing the result of Pigozzi mentioned in the remarks at the end of Section 3. Let δ be the system of definitions for f in $\cdot$ such that fv_0 is defined as $v_0 \cdot v_0$. By Pigozzi's theorem T has the same Turing degree as the theory based on $f^2 v_0 \approx fv_0$. This last theory is easily seen to be decidable; in fact it is

$$\{f^{n+1} v_i \approx f^{m+1} v_i: n, m, i \in \omega\} \cup \{v_i \approx v_i: i \in \omega\} .$$

Example 4.5. Let f and g be two unary operation symbols. The theory based on $\{fv_0 \approx v_0, gv_0 \approx gv_1\}$ is equationally complete and base decidable. In fact, this theory is the only equationally complete based decidable finitely based equational theory in a non-trivial similarity type, up to renaming the operation symbols and including constant symbols.

Proof. Let T be the theory based on $\{fv_0 \approx v_0, gv_0 \approx gv_1\}$. $\varphi \approx \psi \in T$ if and only if g occurs in both φ and ψ or else $V\varphi = V\psi$ and g occurs in neither φ nor ψ. T is therefore decidable and equationally complete.

Claim. Δ is a base for T if and only if

(1) $T \vdash \Delta$.

(2) For some $i \in \omega$ and $n \in \omega \sim 1$, either $v_i \approx f^n v_i \in \Delta$ or $f^n v_i \approx v_i \in \Delta$.

(3) 1 is the greatest common divisor of $\{|n-m|: f^n v_i \approx f^m v_i \in \Delta$ and $i, n, m \in \omega\}$.

(4) For some $\gamma \in \Delta$, $V\gamma_l \neq V\gamma_r$.

(5) For some $\gamma \in \Delta$, g occurs exactly once on one side of γ and either $V\gamma_l \neq V\gamma_r$ or else g occur at least twice on some side of γ.

Proof. Suppose Δ is a base for T. Then $\Delta \vdash fv_0 \approx v_0$. (2) and (3) follow by easy induction on derivations. (1) is immediate. (4) holds since $\Delta \vdash gv_0 \approx gv_1$. (5) holds because otherwise $\langle 3, h, k\rangle \models \Delta$ where h is the identity function on 3 and $k(\ ) = 1$, $k(1) = 0$, $k(0) = 0$.

Conversely, suppose Δ satisfies (1)–(5). By (2) and (3) $\Delta \vdash fv_0 \approx v_0$. Then by (4) and (5) $\Delta \vdash gv_0 \approx gv_1$. So by (1) Δ is a base for T.

The claim is enough to insure that T is base decidable. Let T' be any other equationally complete finitely based theory in f and g. If T' is the constant theory then T' is base undecidable by Theorem 3.18. Otherwise there must be a non-trivial term θ so that $\theta \approx v_0 \in T'$ and by Theorem 3.12 T' is essentially base undecidable. By Corollary 3.20 all other cases follow.

Definition 4.6. Let Δ and Γ be sets of equations. Δ and Γ are *essentially the same* if and only if for every $\epsilon \in \Delta \sim \mathrm{Ta}$ and $\gamma \in \Gamma \sim \mathrm{Ta}$ there are permutations η and π of Va such that

$$\gamma_l[\langle \eta v_i : i \in \omega\rangle] \approx \gamma_r[\langle \eta v_i : i \in \omega\rangle] \in \Delta$$

or else

$$\gamma_r[\langle \eta v_i : i \in \omega\rangle] \approx \gamma_l[\langle \eta v_i : i \in \omega\rangle] \in \Delta$$

and

$$\epsilon_l[\langle \pi v_i : i \in \omega\rangle] \approx \epsilon_r[\langle \pi v_i : i \in \omega\rangle] \in \Gamma$$

or else

$$\epsilon_r[\langle \pi v_i : i \in \omega\rangle] \approx \epsilon_l[\langle \pi v_i : i \in \omega\rangle] \in \Gamma.$$

Two sets of equations are essentially the same if, aside from tautologies, the renaming of variables, and changing the symmetry of their members, they are identical. Γ and Δ are *essentially different* if they are not essentially the same. If T is a finitely based, decidable theory such that any maximal set of pairwise essentially different irredundant bases is recursive, then T must be base decidable. It should also be noted that if Γ is a set of balanced equations, then $\Theta[\Gamma]$ is also a set of balanced equations, as is easily established by induction on derivations.

Theorem 4.7. *If T is a finitely based theory in a finite recursive similarity type and T is a set of balanced equations then T is decidable, ∇T is finite, every set of pairwise essentially different irredundant bases of T is finite, and T is base decidable.*

Proof. It is sufficient to establish that T is decidable and that every set of pairwise essentially different irredundant bases of T is finite. Let θ be any term. Then there are only finitely many terms φ such that $\theta \approx \varphi$ is balanced, since the similarity type is finite. Hence, there are only finitely many one-to-one sequences of such terms. Since T is balanced, $\theta \approx \varphi$ must have a derivation appearing as one of these sequences if $T \vdash \theta \approx \varphi$. Because T is finitely based, there is a recursive procedure which determines if any given finite sequence of terms is a derivation from T. Hence T is decidable.

Let k be the total number of occurrences of variables, constants, and unary operation symbols in some fixed though arbitrary finite base of T. Let Δ be the set of all balanced equations ϵ such that $V\epsilon_l \cup V\epsilon_r \subseteq k$ and such that no variable, constant, or unary operation symbol occurs in ϵ more than $2k$ times. For any term θ let θ^* be term obtained from θ by renaming the variables from left to right so that if n distinct variables occur in θ then $V\theta^* = n$. Evidently $*$ can be defined in such a way as to be recursive. If Σ is a set of equations $\Sigma^* = \{\varphi^* \approx \psi^* : \varphi \approx \psi \in T\}$. Notice that Σ and Σ^* are essentially the same.

Claim. If Σ is a base for T then $\Sigma^* \cap \Delta$ is a base for T.

Proof. Let Γ be a base for T such that $\Gamma \subseteq \Delta$. Let $\varphi \approx \psi \in \Gamma$. Then $\Sigma \vdash \varphi \approx \psi$ and so $\Sigma^* \vdash \varphi \approx \psi$. From the definition of derivation and $\Gamma \subseteq \Delta$ it follows that $\Sigma^* \cap \Delta \vdash \varphi \approx \psi$. Consequently $\Sigma^* \cap \Delta \vdash \Gamma$ and so $\Sigma^* \cap \Delta$ is a base for T.

Let Σ be any irredundant set of equations, then Σ^* is irredundant. By the claim if Σ is an irredundant base for T then $\Sigma^* \subseteq \Delta$. But Δ is finite so there are only finitely many essentially different irredundant bases of T. This proves the theorem.

Observe that the limitations imposed on the occurrence of constants and unary operation symbols by the definition of balanced equations are essential in this theorem. By considering Mal'cev's undecidable theory in Theorem 3.3 or a finitely presented semigroup with an unsolvable word problem it is easy to see that the decidability of T can fail if these limitations are relaxed. On the basis of Theorem 3.12, the base undecidability theorem, it would also be easy to construct examples violating the base decidability of T but which would still be decidable.

The next theorem is based on a suggestion of Ralph McKenzie and serves to show that the connection between ∇T and the base decidability of T is at least not entirely simple.

Theorem 4.8. *If T is a theory of commutative semigroups and $T \nvdash v_0^{n-1} \approx v_1^{n-1}$ and $T \vdash \{v_0 \cdot v_1 \cdot \ldots \cdot v_{n-1} \approx v_n^n, v_0 v_1^{n-2} \approx v_0^2 v_1^{n-3}\}$ for some $n \in \omega \sim 4$ then*

(i) *∇T is finite,*

(ii) *T is finitely based and base decidable,*

(iii) *T has infinitely many essentially different irredundant bases.*

Proof. The proof is presented as a sequence of 12 claims. φ and ψ range over terms. The notion of length is modified so that $L\varphi$ denotes the number of occurrence of variables in φ.

(1) If $T \vdash \varphi \approx \psi$ and $L\varphi < n$ then $L\varphi = L\psi$.

Proof. Observe that $T \vdash v_0^n \approx v_1^m$ whenever $m \geq n$. Recall that $T \nvdash v_0^{n-1} \approx v_1^{n-1}$. Now $T \vdash v_0^{L\varphi} \approx v_0^{L\psi}$ by substituting v_0 for all variables in $\varphi \approx \psi$. Since $L\varphi < n$ it follows $L\psi < n$ and furthermore

$$T \vdash v_0^{L\varphi + n - (L\varphi + 1)} \approx v_0^{L\psi + n - (L\varphi + 1)} .$$

Consequently

$$T \vdash v_0^{n-1} \approx v_0^{n + (L\psi - L\varphi) - 1} .$$

So $L\varphi \geq L\psi$. By the symmetry of the argument $L\varphi = L\psi$.

(2) If $T \vdash \varphi \approx \psi$ and $L\varphi < n$ then $V\varphi \approx V\psi$.

Proof. This is immediate from (1).

(3) If $L\varphi \geq n$ then $T \vdash \varphi \approx \psi$ just in case $L\psi \geq n$.

Proof. This follows from (1) since $T \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n$. Let E be the set of all equations ϵ such that $L\epsilon_l, L\epsilon_r < n$.

(4) If Γ is a base for T then $\Gamma \cap E \vdash T \cap E$.

Proof. From (1) and (2) by a simple induction on derivations.

(5) If Γ is a base for T then there are equations $\epsilon, \gamma \in \Gamma$ such that, up to symmetry and renaming of variables, $\epsilon_l = v_0 \cdot \ldots \cdot v_{n-1}$ and either $L\epsilon_r > n$ or $V\epsilon_l \neq V\epsilon_r$, and $V\gamma_l \neq V\gamma_r$.

Proof. $\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n$. Therefore the existence of such a γ is assured by the definition of derivation. By (1) and (2), if $\delta \in \Gamma$ and $L\delta_l < n$ and there is $\theta \in {}^\omega\text{Te}$ such that $\delta_l[\theta]$ is a subterm of $v_0 \cdot \ldots \cdot v_{n-1}$ then $v_0 \cdot v_1 \approx v_1 \cdot v_0 \vdash \delta$ and $i(\delta_l) \leq 1$ for each $i \in \omega$. Consequently

$\Gamma \cap E \nvdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n$. The existence of the specified ϵ now follows by the definition of derivation.

(6) T is finitely based.

Proof. Due to a theorem in [28] every commutative semigroup is finitely based. However, observe that $(T \cap E) \cup \{v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n\}$ is a base for T by (1), (2), and (3). Let Σ be the set of equations ϵ such that $\epsilon \in T \cap E \sim \mathrm{Ta}$ and $V\epsilon_l \cup V\epsilon_r \subseteq n-1$. Then Σ is finite and Σ is a base for $\Theta[T \cap \Sigma]$ by (1) and (2). Hence $\Sigma \cup \{v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n\}$ is a finite base for T.

(7) If $T \vdash \Gamma$ and Γ is finite then $\{\varphi \approx \psi \colon L\varphi < n \text{ and } \Gamma \vdash \varphi \approx \psi\}$ is recursive.

Proof. Let φ be a term. There are only finitely many terms ψ so that $V\psi = V\varphi$ and $L\varphi = L\psi$. Consequently, there are only finitely many sequences ξ which are one-to-one and such that $L\xi_i = L\varphi$ and $V\xi_i = V\varphi$ for each i in the domain of ξ. If $L\varphi < n$, by (1) and (2) $\Gamma \vdash \varphi \approx \psi$ just in case one of these finitely many sequences is a derivation. Since Γ is finite and the sequence can be recursively constructed from φ, it follows that $\{\varphi \approx \psi \colon L\varphi < n \text{ and } \Gamma \vdash \varphi \approx \psi\}$ is recursive.

(8) T is decidable.

Proof. This follows from (7) and (3) by means of (6).

(9) Γ is a base for T iff $T \vdash \Gamma$, $\Gamma \cap E \vdash T \cap E$, and there are $\epsilon, \gamma \in \Gamma$ as specified by (5).

Proof. The necessity of these conditions follows from (4) and (5). To prove the converse two cases will be considered. Assume Γ fulfills the conditions on the right side of (9). In particular, $\Gamma \vdash \{v_0 \cdot v_1 \approx v_1 \cdot v_0, v_0 \cdot v_1^{n-2} \approx v_0^2 \cdot v_1^{n-3}, (v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\}$.

Case 1. $V\epsilon_l \neq V\epsilon_r$ and $L\epsilon_l = L\epsilon_r$. Since $\Gamma \vdash v_0 \cdot v_1 \approx v_1 \cdot v_0$, it follows that there is $\eta \in {}^n\omega$ such that $\Sigma_{i \in (n-1)} \eta_i = n$ and

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_1^{\eta_1} \cdot \ldots \cdot v_{n-1}^{\eta_{n-1}} . \qquad (*)$$

Assume without loss of generality that $\eta_1 > 2$. Suppose $i \in n \sim 2$ and $\eta_i > 0$. Then for some θ and θ' such that $L\theta = i - 1$,

$$\Gamma \vdash v_1^{\eta_1} \cdot \ldots \cdot v_{n-1}^{\eta_{n-1}} \approx v_i \cdot v_1 \cdot \theta \cdot v_1 \cdot \theta'$$

by the commutative (and associative law). So by (*)

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_1^{\eta'_1} \cdot \ldots \cdot v_{n-1}^{\eta'_{n-1}}$$

where $\eta' \in {}^{n\sim 1}\omega$ such that $\Sigma_{i\in(n\sim 1)} \eta'_i = n$ and $\eta'_1 > \eta_1$. By repeating the process no more than n times, one obtains $\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_1^n$. By the commutative and associative laws $\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_0^n$. Consequently, $\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n$ and Γ is a base for T.

Case II. $\mathrm{V}\epsilon_l = \mathrm{V}\epsilon_r$ or $\mathrm{L}\epsilon_l \neq \mathrm{L}\epsilon_r$. So $\mathrm{L}\epsilon_r > n$. By means of the associative and commutative laws and a suitable substitution instance of ϵ, it follows that there is $\eta \in {}^n(\omega \sim 1)$ such that $\Sigma_{i\in n}\, \eta_i > n$ and

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_0^{\eta_0} \cdot \ldots \cdot v_{n-1}^{\eta_{n-1}} . \tag{**}$$

By means of the associative and commutative laws and (**), after repeated application, it follows that, for $M = \eta_0 \eta_1 \ldots \eta_{n-1}$,

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_0^M \cdot \ldots \cdot v_{n-1}^M .$$

Note that $M > n > 2$. So for any $j \in \omega$

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j} . \tag{***}$$

Assume, without loss of generality that $\mathrm{V}\gamma_l \sim \mathrm{V}\gamma_r \neq 0$. Then

$$\Gamma \vdash v_0^r \approx v_0^p v_1^q ,$$

where $p + q = \mathrm{L}\gamma_l$ and $r = \mathrm{L}\gamma_r$. Let $N = rq$.
Hence

$$\Gamma \vdash v_0^N \approx v_0^{pq} v_1^{pq}$$

and thus

$$\Gamma \vdash v_1^N \approx v_0^{pq} v_1^{pq}$$

by the commutative law. Finally

$$\Gamma \vdash v_0^N \approx v_1^N .$$

Pick j so large that $M^j > N + n$. From (***)

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - N} (v_0 \cdot \ldots \cdot v_{n-1})^N .$$

Therefore from $\Gamma \vdash v_0^N \approx v_1^N$ it follows that

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - N} \cdot v_n^N .$$

Recall that $\Gamma \vdash v_1^{n-2} \cdot v_0 \approx v_1^{n-3} \cdot v_0^2$ and so

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - N + 1} \cdot v_n^{N-1} .$$

Repeat this process $N-2$ times to obtain

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - 1} \cdot v_n . \qquad (****)$$

Moreover $\Gamma \vdash v_0^n \approx v_0^{n(M^j - 1)} \cdot v_n$ by substituting v_0 for $v_0, v_1, \ldots$, and v_{n-1} in (****). Consequently

$$\Gamma \vdash v_n^n \approx v_n^{n(M^j - 1)} \cdot (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - 1} \cdot v_{n+1}$$

by substitution. But also

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx (v_0 \cdot \ldots \cdot v_{n-1})^{M^j - 1} \cdot v_{n+1}$$

by substitution in (****). So at last

$$\Gamma \vdash v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^n$$

and thus Γ is a base for T.

(10) T is base decidable.
Proof. From (9) by means of (8) and (7) and (6).

(11) ∇T is finite.
Proof. Observe that $\Theta[T \cap E]$ has only finitely many essentially different irredundant bases. Let m be a bound on their cardinalities. Then $m + 2$ is a bound on ∇T by (9).

(12) T has infinitely many essentially different irredundant bases.
Proof. Let Δ be an irredundant base for $\Theta[T \cap E]$. For each $m \in \omega$, $\Delta \cup \{v_0 \cdot \ldots \cdot v_{n-1} \approx v_n^{n+m}\}$ is an irredundant base for T and no two of these are essentially the same.

The theorem is established by (11), (6), (10), and (12).

Example 4.9. For $n \in \omega \sim 4$ there is a commutative semigroup $\mathfrak{A}$ such that

(i) $\mathfrak{A}$ has cardinality n,

(ii) Th$\mathfrak{A}$ is finitely based and base decidable,

(iii) ∇ Th$\mathfrak{A}$ is finite,

(iv) Th$\mathfrak{A}$ has infinitely many essentially different irredundant bases.

Proof. Let $\mathfrak{A} = \langle n, \circ \rangle$ where

$$m \circ k = \begin{cases} m+k & \text{if } m+k<n \text{ and } m \neq 0 \text{ and } k \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $m, k \in n$. Then $\mathfrak{A}$ is clearly a commutative semigroup and $\mathfrak{A} \models v_0 \cdot v_1 \cdot \ldots \cdot v_{n-1} \approx v_n^n$ since $m_0 \circ m_1 \circ \ldots \circ m_{n-1} = 0$ for $m \in {}^n n$. Let $a \in {}^{n-1}n$ and observe that

$$a_0 \circ a_1 \circ \ldots \circ a_{n-2} = \begin{cases} n-1 & \text{if } a_i = 1 \text{ for all } i \in n-1, \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathfrak{A} \models v_0 \cdot v_1^{n-2} \approx v_0^2 \cdot v_1^{n-3}$ and $\mathfrak{A} \not\models v_0^{n-1} \approx v_1^{n-1}$. So Th$\mathfrak{A}$ satisfies the hypotheses of Theorem 4.8 and the conclusion follows.

Remark 4.10. In the course of proving Theorem 7 of [36], J. Ng and A. Tarski observed that each of the following theories is base decidable: the theory of all semigroups, the theory of all commutative groupoids, and the theory of all commutative semigroups. Since each of these theories is evidently finitely based and balanced their base decidability follows from Theorem 4.7.

Instead of considering whether or not arbitrary finite sets may be bases for some equational theory it is reasonable to consider only those sets already containing some specified equations. For example, let T be the theory of semi-lattices. It turns out that the family of all bases for T which contain the associative law is recursive, while the family of all bases for T which contain the commutative law is not recursive. The final portion of this section will deal with such relativized notions of base decidability.

Definition 4.11. T is *base decidable modulo* Δ if and only if T is an equational theory in a recursive similarity type, Δ is a set of equations in the same type and $\{\Gamma\colon \Gamma$ is finite and $\Gamma \cup \Delta$ is a base for $T\}$ is a recursive set.

The notions of T being base undecidable modulo Δ and being essentially base undecidable modulo Δ can be introduced in the same way the analogous concepts were introduced in Section 3.

Theorem 4.12. *Let T be the theory based on* $\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0 \cdot v_1 \approx v_1 \cdot v_0, v_0 \cdot v_0 \approx v_0\}$. *($T$ is the theory of semilattices). T is base decidable modulo the associative law.*

Proof. $T = \{\varphi \approx \psi : \varphi$ and ψ are terms and $\mathrm{V}\varphi = \mathrm{V}\psi\}$. Therefore T is decidable. The theorem is an immediate consequence of

Claim. $\Gamma \cup \{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\}$ is a base for T if and only if

(1) $T \vdash \Gamma$,

(2) $\{i(\gamma_l) - i(\gamma_r) : i \in \omega$ and $\gamma \in \Gamma\}$ is relatively prime,

(3) $1 = \min\{\mathrm{L}\gamma : \gamma \in t(\Gamma \sim \mathrm{Ta})\}$,

(4) There are $\epsilon, \gamma \in \Gamma$ such that for some $i, j, k, l \in \omega$, $i \neq j$, $k \neq l$, v_i is the left most symbol in ϵ_l, v_j is the left most symbol in ϵ_r, v_l is the right most symbol in γ_l, and v_k is the right most symbol in γ_r.

Proof. Suppose (1)–(4) hold. By Lemma 4.2

$$\Gamma \cup \{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\} \vdash v_0^2 \approx v_0 .$$

So it is enough to show

$$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0^2 \approx v_0, \epsilon, \gamma\} \vdash v_0 \cdot v_1 \approx v_1 \cdot v_0 .$$

Let

$$\Sigma = \{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2), v_0^2 \approx v_0, \epsilon, \gamma\} .$$

Evidently $\Sigma \vdash \epsilon_l \cdot \gamma_l \approx \epsilon_r \cdot \gamma_r$ and for every $n \in \omega \sim 1$, $\Sigma \vdash v_0^n \approx v_0$. So for some strings (possibly empty) θ and θ' of the variables v_0 and v_1, $\Sigma \vdash v_0 \cdot \theta \cdot v_1 \approx v_1 \cdot \theta' \cdot v_0$. By means of $\Sigma \vdash v_0^n \approx v_0$ any string of identical variables may be replaced by a single occurrence of that variable.

So

$$\Sigma \vdash (v_0 \cdot v_1)^p \approx (v_1 \cdot v_0)^q$$

for some $p, q > 0$. Therefore

$$\Sigma \vdash v_0 \cdot v_1 \approx v_1 \cdot v_0$$

and so

$$\Gamma \cup \{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\}$$

is a base for T.

Conversely, suppose

$$\Gamma \cup \{(v_0 \cdot v_1) \cdot v_2 \approx v_1 \cdot (v_1 \cdot v_2)\}$$

is a base for T.

(1) is immediate. (2) and (3) follow from Lemma 4.2. Let $\mathfrak{A} = \langle 2, \otimes \rangle$ and $\mathfrak{B} = \langle 2, \oplus \rangle$ where $a \otimes b = b$ and $a \oplus b = a$, for all $a, b \in 2$. Then $\mathfrak{A} \not\models v_0 \cdot v_1 \approx v_1 \cdot v_0$ and likewise $\mathfrak{B} \not\models v_0 \cdot v_1 \approx v_1 \cdot v_0$. But one of $\mathfrak{A}$ and $\mathfrak{B}$ is a model of any set of equations for which (4) fails. Hence Γ has to satisfy (4) and the claim is established and with it the theorem.

Theorem 4.13. *The theory of semilattices (in the operation symbol* $\cdot$*) is essentially base undecidable modulo the commutative law.*

Proof. The proof will only be sketched since it is very much like the proof of the base undecidability theorem. In fact, that theorem could be established for these relativized notions provided the relativized notion of jointly universal is used rather than the absolute notion. Of course, the conditions on the non-trivial term will not be so nicely stated once this adaptation is instituted. This is largely due to the need to enhance Theorem 2.33.

By Theorem 3.15 there are four distinct terms δ_0, δ_1, δ_2, and δ_3 in the variable v_0 such that $\{\delta_0, \delta_1, \delta_2, \delta_3\}$ satisfies the subterm condition modulo the commutative law. Let φ, ψ be any non-variable terms in f, g, and v_0. δ is construed as a system of definitions for f, g, h, and k in $\cdot$. Let Γ be a base for T.

Claim. $M \vdash \varphi \approx \psi$ if and only if

$$\begin{aligned} &\mathrm{in}_\delta^* M \cup \{v_0 \cdot v_1 \approx v_1 \cdot v_0\} \\ &\quad \cup \{\delta_2[\mathrm{in}_\delta \varphi[\delta_3[\gamma_l]]] \approx \delta_2[\mathrm{in}_\delta \psi[\delta_3[\gamma_r]]]: \gamma \in \Gamma\} \\ &\quad \cup \{\delta_2[\mathrm{in}_\delta \psi[\delta_3]] \approx v_0\} \end{aligned}$$

is a base for T.

Proof. If $M \vdash \varphi \approx \psi$ then the set on the right, $B(\varphi \approx \psi, \delta, \Gamma) \cup \{v_0 \cdot v_1 \approx v_1 \cdot v_0\}$, is certainly a base for T. (Observe that $T \vdash B(\varphi \approx \psi, \delta, \Gamma)$.) The converse is established just as in the proof of Theorem 3.12. Suppose $M \nvdash \varphi \approx \psi$. Then $\mathrm{in}_\delta^* M \cup \{v_0 \cdot v_1 \approx v_1 \cdot v_0\} \nvdash \mathrm{in}_\delta \varphi \approx \mathrm{in}_\delta \psi$ by the relativized version of the reduction theorem. Let $\mathfrak{A}$ be a denumerably infinite algebra such that $\mathrm{Th}\,\mathfrak{A} = \Theta[\mathrm{in}_\delta^* M]$. By Lemma 3.8 there is another denumerably infinite algebra $\mathfrak{B}$ so that $\mathfrak{B} \models \mathrm{in}_\delta^* M$ and $\mathfrak{B}$ invalidates $\mathrm{in}_\delta\, \varphi \approx \mathrm{in}_\delta\, \psi$ infinitely often (see Lemma 3.8 for precision). Finally, since $\{\delta_0, \delta_1, \delta_2, \delta_3\}$ is jointly ω universal modulo the commutative law, there is an algebra $\mathfrak{C}$ with universe B such that $\mathfrak{C} \models B(\varphi \approx \psi, \delta, \Gamma) \cup \{v_0 \cdot v_1 \approx v_1 \cdot v_0\}$ and $\delta_2^{\mathfrak{C}} = \delta_2^{\mathfrak{B}} \neq \delta_3^{\mathfrak{B}} = \delta_3^{\mathfrak{C}}$. Consequently, $\mathfrak{C} \not\models T$ and $B(\varphi \approx \psi, \delta, \Gamma) \cup \{v_0 \cdot v_1 \approx v_1 \cdot v_0\}$ is not a base for T. In this way the claim is proven.

Evidently, by the claim, T is base undecidable modulo the commutative law. To see that it is essentially undecidable modulo the commutative law let $T' \supseteq T$ and T' be a finitely based equational theory. It is easy to see that the claim is going to hold for T' as well as T. So the theorem is established.

It should be noted that the proof sketched above used almost no explicit information about T. In fact, most of the theorems of Section 3 have analogs modulo the commutative law. For example, the theory of abelian groups is also essentially base undecidable modulo the commutative law. In consequence, this also applies to all finitely based theories of commutative groups and rings, to theories of lattices, and to theories of Boolean algebras.

Theorem 4.14. *Let T be any finitely based undecidable equational theory and $\Delta \subseteq T$. T is a base undecidable modulo Δ.*

A finitely based undecidable theory of semigroups is presented in [25]. By Theorem 4.14 this theory must be base undecidable modulo the associative law.

The notion of compatibility decidable set of equations is closely connected with base decidability modulo a set of equations.

Definition 4.15. Let σ be a recursive similarity type. A set $\Delta \subseteq \mathrm{Eq}_\sigma$ is called *compatibility decidable* if and only if Eq_σ is base decidable modulo Δ.

P. Perkins, in his doctoral thesis, showed that the theory of semigroups is compatibility decidable. It turns out that, by various simple constructions, this also holds true for the theory of groups, the theory of abelian groups, the theory of rings, the theory of lattices, and some other common equational theories. This means it is possible, at least in principle to determine in each particular case whether a finite set of equations is true in some non-trivial group, abelian group, ring, lattice, etc.

Lemma 4.16. *If T is a compatibility decidable equational theory of similarity type σ then every finitely based extension of T in similarity type σ is compatibility decidable.*

Lemma 4.17. *Let σ and τ be finite recursive similarity types. Let T be an equational theory of type σ and T' be an equational theory of type τ such that T and T' are definitionally equivalent. T is compatibility decidable if and only if T' is compatibility decidable.*

Theorem 4.18. *Each of the following theories is compatibility decidable provided it is formulated in a finite recursive similarity type:*

(1) *(P. Perkins [28]) The theory of semigroups;*

(2) *The theory of groups;*

(3) *The theory of rings;*

(4) *The theory of lattices;*

(5) *The theory of Boolean algebras.*

Proof. The general idea of the proof is the same in all cases. First it is shown that each equationally complete theory, in every case, is the theory of some finite algebra. It is then established, in every case, that for any finite set of equations a number k can be effectively found so that if the set of equations is true in any non-trivial model then it is true in a model of one of the complete theories of cardinality less than k.

(1) *Semigroups.* Equationally complete semigroups were classified in [14]. They are: the theories of (i) the constant semigroup, (ii) semilattices, (iii) left multiplication, i.e. $x \cdot y \approx y$, and right multiplication semigroups, and (iv) cyclic groups of order p, for every prime p.

Let Γ be a finite set of equations. Since (i)–(iii) are all theories of two element algebras it is simple to check if Γ is true in one of them.

Now

$$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\} \cup \Gamma \vdash R\Gamma .$$

Pick m and r as in Lemma 4.2. Then

$$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2)\} \cup \Gamma \vdash v_0^m \approx v_0^{m+r} .$$

If Γ fails to hold in the two element constant semigroup then

$$\{(v_0 \cdot v_1) \cdot v_2 \approx v_0 \cdot (v_1 \cdot v_2),\ v_0 \approx v_0^r\}$$

must be consistent if Γ is compatible with the associative law. Now if

$p > r$ and p is prime then $v_0 \approx v_0'$ fails to hold in the cyclic group of order p. So Γ is true in some semigroup if and only if Γ is true in one of the algebras in (i)–(iii) or in some cyclic form of prime order no greater than r. Hence the theory of semigroups is compatibility decidable.

(2) *Groups.* The equationally complete theories of groups are easily seen to be the theories of cyclic groups of prime order. Let Γ be a finite set of equations in $\cdot$ and $^{-1}$. Γ is compatible with group theory if and only if it is compatible with the theory of Abelian groups. With the help of the group axioms and the commutative law Γ is equivalent to a set Γ' of equations in just $\cdot$. Now use case (1) to decide whether Γ' holds in a cyclic group of prime order. Hence the theory of groups is compatibility decidable.

(3) *Rings.* Equationally complete rings were classified in [35]. They are just the theories of p-rings for each prime p (the ring of integers modulo p) and the theories of p-zero rings (i.e. rings whose multiplication is constantly zero and whose additive group is the cyclic group of order p).

Let Γ be any finite set of equations in $\cdot$, $+$, and the unary operation $-$. Γ is compatible with ring theory if and only if it is compatible with the theory of commutative rings. With the help of the axioms of commutative ring theory Γ is equivalent to a set Γ' in which each equation is in only the operation symbols $\cdot$ and $+$. If Γ' is a set of tautologies then Γ is true in every commutative ring. Otherwise Γ' fails in the ring of integers and in fact in the semiring of natural numbers. Hence, there is an assignment of natural numbers to the variables in Γ' so that some equation in Γ' fails. This assignment can, moreover, be effectively found since Γ' is finite. Pick n so large that the values of all polynomials used to evaluate Γ' from this assignment are less than n. If $p > n$ and p is prime then Γ' fails in the ring of integers modulo p. If Γ' holds in the ring of integers modulo q, for some prime $q < n$, then Γ is compatible with ring theory. Otherwise Γ and, hence Γ', holds in no p-ring. Thus if Γ is compatible with the theory of rings then Γ holds in some p-zero ring. Hence Γ' may be reduced yet again to Γ'' by eliminating all occurrences of $\cdot$. In this way Γ'' becomes a set of semigroup equations that can be handled as in (1). Therefore the theory of rings is compatibility decidable.

(4) and (5) *Lattices and Boolean algebras*. The only equationally complete theories in these cases are the theories of the two element algebras. Consequently, these cases are immediate.

Problems raised by Section 4

1. If T is a finitely based equational theory in a recursive similarity type and T is finite, is T base decidable?

2. Is there a finitely based decidable theory of semigroups which is base undecidable modulo the associative law?

5. Acknowledgements

This paper is a large part of my Ph. D. thesis submitted to the University of California at Berkeley in June 1972. It is with special pleasure that I acknowledge the help and encouragement which many have given me throughout the research and preparation of this paper. Professor Alfred Tarski, who guided this research, was a constant source of encouragement, questions, and ideas. Tarski's concern for me has extended beyond purely mathematical and even purely intellectual considerations and I am pleased to record my appreciation here. Professor Ralph McKenzie had a major influence on this paper and at several points, which were indicated in the body of the paper, he supplied key ideas. Professors Walter Taylor, Jan Mycielski, Don Pigozzi, William Craig, and Leon Henkin each provided firm criticism about the style and content of this paper and I am grateful to each of them. In particular Professor Pigozzi pointed out several mathematical errors in earlier versions of the paper and Professor Mycielski supplied some valuable references and historical connections. Roger Maddux devoted an incredible amount of effort to checking all the mathematical details in an earlier manuscript. Elizabeth and Lonnie Nesseler, Lonnie Jr., and Marsha Taft each gave me important personal support while I was in the midst of research and without which I would have completed this paper with difficulty. I am grateful to Miss D. Gessner for the careful typing of this long paper

Financial support for the research of this paper came from National Science Foundation grant GP-6232X, which funded a research project in the foundation of mathematics directed by Professors Leon Henkin and Alfred Tarski, and also from NSF graduate fellowships. The writing of this paper was accomplished with the support of the National Research Council of Canada and Dartmouth College.

References

[1] G. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Philos. Soc. 31 (1935) 433–454.
[2] A.D. Bol'bot, A mnogoobrazijach Ω-algebr, Algebra i Logika 9 (1970) 406–414.
[3] W.W. Boone, Certain simple unsolvable problems of group theory, Indag. Math. 16 (1954) 231–237.
[4] W.W. Boone, Word problems and recursively enumerable degrees of unsolvability, A sequel on finitely presented groups, Ann. of Math. 84 (1966) 49–84.
[5] S. Burris, On the structure of the lattice of equational classes $\mathcal{L}(\tau)$, Algebra Universalis 1 (1971) 39–45.
[6] W. Felscher, Equational maps, in: K. Schütte et al., ed., Contributions to Mathematical Logic (North-Holland, Amsterdam 1968).
[7] G. Grätzer, Universal Algebra (D. Van Nostrand, San Francisco 1968).
[8] M. Hall, The word problem for semigroups with two generators, J. Symbolic Logic 14 (1949) 115–118.
[9] L. Henkin, J.D. Monk, A. Tarski, Cylindric Algebras I (North-Holland, Amsterdam 1971).
[10] J.R. Isbell, On the problem of universal terms, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966) 593–595.
[11] J. Jezek, On atoms in lattices of primitive classes, Commentationes Mathematicae Universitatis Carolinae 11 (1970) 515–532.
[12] J. Jezek, The existence of upper semicompliments in lattices of primitive classes, Commentationes Mathematicae Universitatis Carolinae 12 (1971) 519–532.
[13] J. Kalicki, The number of equationally complete classes of equations, Indag. Math. 17 (1955) 660–662.
[14] J. Kalicki, D. Scott, Equational completeness of abstract algebras, Indag. Math. 17 (1955) 650–659.
[15] J. Lukasiewicz, A. Tarski, Untersuchungen über den Aussagenkalkül, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Class III, Sciences Mathematiques et Physiques 23 (1930) 30–50.
[16] A.I. Mal'cev, Structural characteristics of certain classes of algebras, (Russian) Doklady Akad. Nauk. SSSR 120 (1958) 29–32.
[17] A.I. Mal'cev, Identical relations on varieties of quasigroups (Russian), Mat. Sb. 69 (1966) 3–12. [Translations, AMS Translations, series 2, 82 (1968) 225–235].
[18] A.A. Markov, On the impossibility of certain algorithms in the theory of associative systems, (Russian), Doklady Akad. Nauk. SSSR n.s. 55 (1947) 587–590.
[19] G. McNulty, The problem of base decidability for equational theories of algebras, Notices of the AMS 17 (1970) 638.
[20] G. McNulty, On algebras whose equational theories are essentially base undecidable, Notices of the AMS 17 (1970) 657.
[21] G. McNulty, Base decidable theories and irredundant bases, Notices of the AMS 18 (1971) 1108.
[22] G. McNulty, Undecidable properties of finite sets of equations, Notices of the AMS 19 (1972) A-328.
[23] G. McNulty, Undecidable properties of finite sets of equations, to appear in J. Symb. Logic.
[24] G. McNulty, Structural diversity in the lattice of equational theories, to appear.
[25] V.L. Murskiĭ, Some examples of varieties of semigroups, (Russian), Mat. Zametki 3 (1968) 663–670.
[26] V.L. Murskiĭ, Nondiscernible properties of finite systems of identity relations, (Russian), Doklady Akad. Nauk. SSSR 196 (1971) 520–522. [Translation: Soviet Math. Dokl. 12 (1971) 183–186.]

[27] P.S. Novikov, On the algorithmic undecidability of the word problem in groups, (Russian), Trudy Math. Inst. Steklov no. 14 Izdat Akad. Nauk. SSSR, Moscow, 1955.
[28] P. Perkins, Decision problems for equational theories of semigroups and general algebras, Ph. D. thesis, Univ. of California, Berkeley, Calif. 1966.
[29] P. Perkins, Unsolvable problems for equational theories, Notre Dame Journal of Formal Logic 8 (1967) 175–185.
[30] R. Platek, Ph. D. Thesis, Stanford University 1966.
[31] E.L. Post, Recursive unsolvability of a problem of Thue, J. Symbolic Logic 12 (1947) 1–11.
[32] J. Robinson, Recursive functions of one variable, Proc. Am. Math. Soc. 19 (1968) 815–820.
[33] H. Rogers, Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).
[34] A. Tarski, An undecidable system of sentential calculus, J. Symbolic Logic 18 (1953) 189.
[35] A. Tarski, Equationally complete rings and relation algebras, Indag. Math. 18 (1956) 39–46.
[36] A. Tarski, Equational logic, in: K. Schütte et al., eds., Contributions to Mathematical Logic (North-Holland, Amsterdam, 1968).