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UNDECIDABLE PROPERTIES OF FINITE SETS OF EQUATIONS

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In memory of Robert Louis Eldridge

§0. Introduction. Though equations are among the simplest sentences available in a first order language, many of the most familiar notions from algebra can be expressed by sets of equations. It is the task of this paper to expose techniques and theorems that can be used to establish that many collections of finite sets of equations characterized by common algebraic or logical properties fail to be recursive. The following theorem is typical.

THEOREM. *In a language provided with an operation symbol of rank at least two, the collection of finite irredundant sets of equations is not recursive.*

Theorems of this kind are part of a pattern of research into decision problems in equational logic. This pattern finds its origins in the works of Markov [8] and Post [20] and in Tarski's development of the theory of relation algebras; see Chin [1], Chin and Tarski [2], and Tarski [23]. The papers of Mal'cev [7] and Perkins [16] are more directly connected with the present paper, which includes generalization of much of Perkins' work as well as extensions of a theorem of D. Smith [22]. V. L. Murskiĭ [14] contains some of the results below discovered independently. Not all known results concerning undecidable properties of finite sets of equations seem to be susceptible to the methods presented here. R. McKenzie, for example, shows in [9] that for a language with an operation symbol of rank at least two, the collection of finite sets of equations with nontrivial finite models is not recursive. D. Pigozzi has extended and elaborated the techniques of this paper in [17], [18], and [19] to obtain new results concerning undecidable properties, particularly those of algebraic character.

This paper is itself a continuation of [13] where the basic methods used here were developed. The present paper includes without proofs the pertinent results of the earlier paper.

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[1] I would like to thank the referee for his patient reading of an earlier version of this paper and for pointing out several misstatements in that version.

was my thesis advisor and I am grateful for his advice and encouragement. I also found special profit from talks with Professors Ralph McKenzie and Don Pigozzi. Some of the results below were announced in [10], [11], and [12].

§1 gathers together those results from [13] which are essential in the remainder of the paper; the section also contains several new theorems useful in §2. Unfortunately §1 is rather technical and the interested reader is encouraged to consult [13] especially concerning the proof of Theorem 1.7. §2 contains six theorems. The first three are general undecidability results with relatively simple proofs. The others have a more specialized character and the last two have more delicate proofs. It has turned out to be unwieldy to present proofs of all the undecidability results accessible by our methods. Instead it is hoped that the reader can reconstruct the remaining proofs.

Our general references in algebra are Gratzer [3] and Henkin, Monk, and Tarski [4, Chapter 0]. The reader is assumed to be familiar with basic notions from universal algebra such as *congruence lattices* and *subdirectly irreducible algebras.* Chang and Keisler [0] is our principal reference for notions and notation from model theory and first order logic. Tarski's survey article [24] is our reference for concepts specific to equational logic. The reader unfamiliar with undecidable theories and recursive functions is referred to Rogers [21].

The remainder of this introduction is a summary of our results.

Let Δ be a set of equations. Δ is *consistent* provided Δ has an infinite model; Δ is *equationally complete* if Δ is consistent and the same equations hold in any two nontrivial models of Δ; Δ is *irredundant* if $\Delta \sim \{\delta\} \nvdash \delta$ for all $\delta \in \Delta$; Δ is κ*-categorical* provided Δ is consistent and any two models of Δ of cardinality κ are isomorphic; Δ is *decidable* if $\{\delta : \Delta \vdash \delta\}$ is recursive; Δ is a *base* of T provided T is a set of equations and Δ and T have the same models; Δ is *essentially finitely based* provided every extension of Δ is finitely based; Δ is *residually small* if there is a cardinal which is an upper bound on the size of the subdirectly irreducible models of Δ; Δ is *residually finite* if all subdirectly irreducible models of Δ are finite; γ is a *nontrivial congruence lattice identity* provided γ is a lattice identity which fails in the lattice of congruences of some algebra; Δ *satisfies the congruence lattice identity* γ if γ is true in the lattice of congruences of every model of Δ. $\nabla\Delta = \{n : \Delta$ has an irredundant base of n equations$\}$.

What follows is a table of undecidable properties of finite sets of equations which have been established by the methods described below. Various weak conditions are sometimes imposed on the language and these conditions are described in §1. If $P(\Sigma)$ is a property of the sets Σ of equations and L is a language, then the corresponding line in the table means that $\{\Sigma : P(\Sigma)$ and Σ is a finite set of L-equations$\}$ is not recursive. I have tried to cite the literature and give credit to the people who discovered the various results. Where one of the results is proved in the body of this paper the appropriate theorem is cited by number—special cases and immediate corollaries are treated similarly.

UNDECIDABLE PROPERTIES

Property of Σ	Conditions on L	References
1. Σ is consistent.	L is nontrivial.	Perkins [16] and McNulty [13].
2. Σ is equationally complete.	L is nontrivial and finite.	Special case: Perkins [16].
3. Σ is decidable.	L is nontrivial.	Theorem 2.6, special case: Perkins [16].
4. Σ is the base of a finite algebra.	L is nontrivial.	Special case: Perkins [16].
5. Σ is consistent and decidable.	L is nontrivial.	Theorem 2.6.
6. Fix a nonempty set S of positive integers. Σ is a base for an algebra with cardinality in S.	L is nontrivial and finite.	Closely related to Theorem 2.6.
7. Σ is ω-categorical.	L is nontrivial and finite.	Closely related to Theorem 2.6.
8. Σ is ω_1-categorical.	L is nontrivial.	Closely related to Theorem 2.6.
9. Σ is categorical in all infinite powers.	L is nontrivial and finite.	Closely related to Theorem 2.6.
10. $\|\Sigma\| = 1$ and Σ is consistent.	L is strong.	Corollary of Theorem 2.1, cf. remark following Theorem 2.5.
11. $\|\Sigma\| = 1$ and Σ is decidable.	L is strong.	Corollary of Theorem 2.4.
12. $\|\Sigma\| = 1$ and Σ is a base of a finite algebra.	L is strong.	Corollary of Theorem 2.4.
13. $\|\Sigma\| = 1$ and Σ is essentially undecidable.	L is strong.	Corollary of Theorem 2.4.

14. Σ is irredundant.	L is nontrivial.	Theorem 2.7.
15. Σ is consistent and irredundant.	L is nontrivial.	Theorem 2.7.
16. $n \in \nabla\Sigma$, where n is a fixed positive integer.	L is nontrivial.	Pigozzi (unpublished) Theorem 2.7, special case: Smith [22].
17. $n \in \nabla\Sigma$ and Σ is consistent, where n is a fixed positive integer.	L is nontrivial.	Theorem 2.7.
18. Σ is essentially finitely based.	L is nontrivial.	Murskiĭ [14].
19. Σ is consistent and essentially finitely based.	L is nontrivial.	
20. Fix Δ a set of L-equations not true in every algebra. $\Sigma \vdash \Delta$.	L is nontrivial.	Like Theorem 2.8, Murskiĭ [14], independently.
21. Fix Δ a set of L-equations with a consistent finitely based extension. Σ is consistent and $\Sigma \vdash \Delta$.	L is strong or L has three unary operation symbols.	Theorem 2.8.
22. Σ is residually small.	L is strong.	Corollary to Theorem 2.4.
23. Σ is residually finite.	L is strong.	Corollary to Theorem 2.4.
24. Σ has arbitrarily large simple models.	L is strong.	Corollary to Theorem 2.4.
25. Σ has no infinite Jonsson model.	L is strong.	Corollary to Theorem 2.4.
26. Fix γ a nontrivial congruence lattice identity. Σ satisfies the congruence lattice identity γ.	L is strong.	Corollary to Theorem 2.5.
27. Σ is a base of a primal algebra.	L is strong.	Corollary to Theorem 2.5.

28. Σ is a base of a quasi-primal algebra.	L is strong.	Corollary to Theorem 2.5.
29. Fix γ a nontrivial congruence lattice identity. Σ is consistent and satisfies the congruence lattice identity γ.	L is strong.	Corollary to Theorem 2.5.
30. Fix Δ a finite set of equations such that $\Delta \vdash \theta \approx v_0$ for some nontrivial term θ. Σ is a base for Δ.	L is nontrivial.	McNulty [13] and Murskiĭ [14], independently.

§1. **Preliminaries.** Algebras are denoted by capital German letters $\mathfrak{A}, \mathfrak{B}, \ldots$ while their universes are denoted by the corresponding Roman capitals $A, B, \ldots$. We will be particularly concerned with the manipulation of equations and terms. Generally lower case Greek letters $\phi, \psi, \sigma, \tau, \theta, \ldots$ represent terms while $\phi \approx \psi$ is an equation between terms. Since on all but a single occasion in this paper every first order sentence is universal, we suppress all quantifiers. Hence $\phi \approx \psi$, unless otherwise specified, is the universal closure of the equation between the terms ϕ and ψ. The letters Γ and Σ are reserved for *finite* sets of equations. If Δ and Θ are sets of equations, then $\Delta \equiv \Theta$ means they have the same models.

The countable first order languages we deal with here all have the same denumerably infinite collection of variables $v_0, v_1, v_2, \cdots$, but they are allowed to differ in their operation symbols. No language occurring in this paper has relation symbols. Each operation symbol has a *rank* which is the number of its arguments. A language is *nontrivial* provided it has an operation symbol of rank at least two or it has at least two different operation symbols of rank one. A language is *strong* if it has an operation symbol of rank at least two. A term θ is *nontrivial* (*strong*) provided a variable occurs in θ and any language for which θ is a term is nontrivial (strong). Terms are conceived as certain strings of variables and operation symbols.

If ϕ is a term in the variables $v_0, \cdots, v_{n-1}$, then $\phi[\theta_0, \cdots, \theta_{n-1}]$ is the term resulting from substituting the terms θ_i for v_i in ϕ for all $i = 0, \cdots, n-1$. If x is the only variable to occur in ϕ, then $\phi[\theta]$ is the result of substituting θ for x. More generally if $\theta = \langle \theta_i : i \in \omega \rangle$ then $\phi[\theta]$ is the term resulting from the substitution of θ_i for v_i in ϕ for all $i \in \omega$. $\phi[\theta]$ (and $\phi[\theta_0, \cdots, \theta_{n-1}]$) are called *substitution instances of* ϕ. We write $\phi \perp \psi$ when ϕ and ψ are terms without common substitution instances; $\phi \top \psi$ means that some substitution instance of

ϕ is also a substitution instance of ψ. If Δ and Φ are sets of terms, then $\Delta \perp \Phi$ means $\delta \perp \phi$ for all $\delta \in \Delta$, $\phi \in \Phi$.

DEFINITION 1.1. A set Δ of terms *satisfies the subterm condition* provided

(i) no variable belongs to Δ and

(ii) if δ, $\delta' \in \Delta$ and ϕ is a subterm of δ which is not a variable such that $\phi \top \delta'$, then $\phi = \delta = \delta'$.

The subterm condition turns out to be a primary tool in the proofs below. It was devised by Ralph McKenzie and introduced in McNulty [13]. Pigozzi uses it in [17], [18], and [19] where sets satisfying the subterm condition are said to be *nonoverlapping*. Some fundamental results concerning the subterm condition which are required in this paper were established in [13]. For convenience they are repeated here without proof as are a few other necessary theorems from the literature.

DEFINITION 1.2. Let Δ be a set of terms and A be any nonempty set. A function F with domain Δ and range included in the set of finitary operations on A is said to agree *according to rank* provided the rank of $F(\delta)$ is the number of distinct variables occurring in δ, for every $\delta \in \Delta$. An element $b \in A$ *annihilates* F provided the value of $F(\delta)$ at any tuple including b is just b, for all $\delta \in \Delta$.

The next theorem is our basic "model-construction" device which is used repeatedly below. It is an extension of Theorem 2.5 from [13] and may be proved in the same manner.

THEOREM 1.3. *Let Δ be any set of terms satisfying the subterm condition; let F be any assignment of finitary functions on ω to the terms in Δ which agrees according to rank. There is an algebra $\mathfrak{A}$ with universe ω such that for all $\delta \in \Delta$ and all terms ϕ, ψ, and π where ϕ and ψ are shorter (as strings of symbols) than any term in Δ*

(i) $\delta^{\mathfrak{A}} = F(\delta)$,

(ii) $\mathfrak{A} \nvDash \phi \approx \psi$ *if* $\phi \neq \psi$,

(iii) *if some member of ω annihilates F, then $\mathfrak{A} \models \phi \approx \pi$ only if every variable occurring in π also occurs in ϕ, and*

(iv) *if ρ and η are terms with $\rho \perp \eta$, $\rho' \perp \Delta$, and $\eta' \perp \Delta$ for all nonvariable subterms ρ' of ρ and η' of η, then $\rho^{\mathfrak{A}}$ and $\eta^{\mathfrak{A}}$ have disjoint ranges; moreover if only one variable occurs in ρ then $\rho^{\mathfrak{A}}$ has no fixed points.*

As will be observed in some of the arguments in the next section we will often find it desirable to translate from one language to another by means of a system of definitions. We assure the faithfulness of the translation by selecting a set of defining terms which satisfies the subterm condition. The next few definitions specify the translation device we use.

DEFINITION 1.4. Consider two languages, L and L', and a one-to-one function δ mapping the operation symbols of L to terms in L' in such a way that $\delta(Q)$ is an L'-term in which $v_0, \cdots, v_{n-1}$ are exactly the variables to occur, whenever Q is an operation symbol of L of rank n. We define the function in_δ, *the interpretation of L in L' on the basis of δ*, from the set of L-terms into the set of L'-terms by the following recursion:

(a) $\mathrm{in}_\delta x = x$ for all variables x,

(b) $\mathrm{in}_\delta Q = \delta(Q)$ for every L-operation symbol Q of rank 0,

(c) $\mathrm{in}_\delta(Q\phi_0 \cdots \phi_{n-1}) = \delta(Q)[\mathrm{in}_\delta\phi_0, \cdots, \mathrm{in}_\delta\phi_{n-1}]$ where Q is an L-operation symbol of rank $n > 0$ and $\phi_0, \cdots, \phi_{n-1}$ are L-terms.

If Φ is a set of L terms then $\mathrm{in}_\delta\Phi = \{\mathrm{in}_\delta\phi : \phi \in \Phi\}$. Likewise if N is a set of L-equations, $\mathrm{in}_\delta N = \{\mathrm{in}_\delta\phi \approx \mathrm{in}_\delta\psi : \phi \approx \psi \in N\}$.

The language in which the key undecidability result, due to Mal'cev, is formulated has two operation symbols and both are unary. For technical reasons it is more convenient to reserve the four letters f, g, h, and k to be unary operation symbols and L_0 to be the language with exactly these operation symbols.

DEFINITION 1.5. Let L and L' be languages with L an expansion of L_0 by unary operation symbols. Let N be any set of L-equations and Θ be any set of L' equations; let δ be a function fulfilling the conditions in Definition 1.4. Finally let $\phi \approx \psi$ be any L-equation. We define:

$$\begin{aligned} B(N, \phi \approx \psi, \delta, \Theta) = {} & \mathrm{in}_\delta N \cup \{\mathrm{in}_\delta(h\phi[kv_0])[\gamma] \approx \mathrm{in}_\delta(h\phi[kv_0])[\rho] : \gamma \approx \rho \in \Theta\} \\ & \cup \{\mathrm{in}_\delta(h\psi[kv_0])[\gamma] \approx \gamma : \text{there exists an } L'\text{-term } \rho \text{ with} \\ & \qquad \rho \approx \gamma \in \Theta \quad \text{or} \quad \gamma \approx \rho \in \Theta\}. \end{aligned}$$

The idea behind this rather complicated definition is to link $N \vdash \phi \approx \psi$ with $\Theta \equiv B(N, \phi \approx \psi, \delta, \Theta)$ at least under favorable circumstances. One of these favorable circumstances is formalized in the following definition.

DEFINITION 1.6. Δ *absorbs* Φ *for* Θ provided Δ and Φ are sets of terms and Θ is a set of equations with

$$\Theta \vdash \{\phi[\delta, \delta, \delta, \cdots] \approx \delta : \delta \in \Delta \quad \text{and } \phi \in \Phi\}.$$

We write "δ absorbs ϕ for Θ" instead of "$\{\delta\}$ absorbs $\{\phi\}$ for Θ".

THEOREM 1.7. (*See Theorem* 3.11 *and Corollary* 2.7 *in* [13].) *Let L and L' be languages with L an expansion of L_0 by unary operation symbols. Let δ be a function satisfying the conditions of Definition* 1.4. *Let Δ be the range of δ, N be a set of L-equations in which v_0 is the only variable to occur, and Θ be a set of L'-equations such that*

(1) *Δ satisfies the subterm condition,*

(2) *$\Theta \vdash \phi \approx \psi$ for all $\phi, \psi \in \Delta$,*

(3) *$\Delta \cup \{\gamma$: there is ρ with $\gamma \approx \rho \in \Theta$ or $\rho \approx \gamma \in \Theta\}$ absorbs Δ for Θ.*

Then for any L-equations μ and η in which v_0 is the only variable to occur we conclude

(4) *$N \vdash \mu$ iff $B(N, \mu, \delta, \Theta) \equiv \Theta$, and*

(5) *if $N \nvdash \mu$, then $N \vdash \eta$ iff $B(N, \mu, \delta, \Theta) \vdash \mathrm{in}_\delta \eta$.*

REMARK. We note here that in proving this theorem in the cases when $N \nvdash \mu$ we provide a model of $B(N, \mu, \delta, \Theta)$ by means of Theorem 1.3. Consequently, a stronger statement based on 1.3(ii)–(iv) is possible and will in fact be tacitly used below.

THEOREM 1.8 (MAL'CEV [7]). *There is a finite set M of equations in which the only variable to occur is v_0 and the only operations symbols to occur are f and g such that*

(1) *if $M \vdash \phi \approx \psi$ and ϕ is a variable, then $\phi = \psi$, and*

(2) $\{\phi \approx \psi: M \vdash \phi \approx \psi$ *and* f, g, *and* v_0 *are the only symbols to occur in* ϕ *and* $\psi\}$ *is not recursive.*

Theorem 1.8 is the basis upon which all the undecidability results of this paper are established. The letter M is reserved throughout this paper for a fixed set satisfying this theorem. We also assume that M is irredundant, i.e. $M \sim \{\mu\} \nvdash \mu$ for any $\mu \in M$. Most of our results proceed from Theorem 1.7 by letting $M = N$ and making appropriate choices for δ and Θ. So most of the rest of this section is devoted to the remaining conditions in Theorem 1.7: constructing sets satisfying the subterm condition which at the same time enjoy some absorption properties. We begin with an artificial though convenient definition.

DEFINITION 1.9. Let θ be a nontrivial term.

(i) If $\theta = p^{k+1}qHp^nx$, where p and q are distinct unary operation symbols, H is a (possibly empty) string of unary operation symbols, x is a variable, and $n, k \in \omega$, then $m(\theta) = \{p^{k+1}qHx, p^kqHx\}$.

(ii) If $\theta = HQ\phi_0 \cdots \phi_{n-1}$, where Q is an operation symbol of rank $n > 1$, H is a (possibly empty) string of unary operation symbols, and $\phi_0, \cdots, \phi_{n-1}$ are terms, then $m(\theta) = \{\theta, \phi_0, \cdots, \phi_{n-1}\}$.

THEOREM 1.10 (THEOREM 2.26 IN [13]). *Let θ be any nontrivial term in which all operation symbols to occur are unary. There is a set Δ of terms such that for any set Θ of equations and any set Φ of terms*

(1) *Δ is infinite,*

(2) *Δ satisfies the subterm condition, and*

(3) *if $\Phi \cup m(\theta)$ absorbs θ for Θ, then $\Phi \cup \Delta$ absorbs Δ for Θ and* $\Theta \vdash \{\phi \approx \psi: \phi, \psi \in \Delta\}$.

THEOREM 1.11 (THEOREM 2.30 IN [13]). Let $\theta = Q\phi_0 \cdots \phi_{n-1}$ *where Q is an operation symbol of rank $n > 1$ and $\phi_0, \cdots, \phi_{n-1}$ are terms. Suppose that the variable x occurs in θ. There is a set Δ of terms such that for any set Θ of equations and any set Φ of terms*

(1) *Δ is infinite,*

(2) *Δ satisfies the subterm condition, and*

(3) *if $\Phi \cup m(\theta)$ absorbs θ for Θ, then $\Phi \cup \Delta$ absorbs Δ for Θ and* $\Theta \vdash \{\phi \approx \theta[x, x, x, \ldots]: \phi \in \Delta\}$.

Yet another definition is helpful in handling the remaining case of this sort.

DEFINITION 1.12. Let θ be a term. θ^+ is defined by the following recursion:

(i) $x^+ = x$ for all variables x,

(ii) $Q^+ = Q$ for all operation symbols Q of rank 0,

(iii) $(p\phi)^+ = \phi^+$ for all unary operation symbols p and terms ϕ,

(iv) $(Q\phi_0 \cdots \phi_{n-1})^+ = Q\phi_0^+ \cdots \phi_{n-1}^+$ for all operation symbols Q of rank $n > 1$ and all terms $\phi_0, \cdots, \phi_{n-1}$.

θ^+ is obtained by deleting all unary operation symbols. $\Delta^+ = \{\delta^+: \delta \in \Delta\}$ whenever Δ is a set of terms.

THEOREM 1.13 (THEOREM 2.33 IN [13]). *Let θ be any nontrivial term. There is a set Δ of terms such that for any set Θ of equations and any set Φ of terms*

(1) *Δ is infinite,*

(2) *if an operation symbol with rank at least two occurs in θ, then Δ^+ is infinite*

and satisfies the subterm condition, and

(3) *if* $\Phi \cup m(\theta)$ *absorbs* θ *for* Θ, *then* $\Phi \cup \Delta$ *absorbs* Δ *for* Θ *and* $\Theta \vdash \{\psi \approx \phi : \psi, \phi \in \Delta\}$.

We need one more sequence of preliminary results on the construction of sets satisfying the subterm condition.

THEOREM 1.14 (THEOREM 2.23 IN [13]). *Let* L *and* L' *be any two languages and let* in_δ *be an interpretation of* L *in* L' *on the basis of* δ. *Let* Φ *be any set of* L *terms such that both* Φ *and the range of* δ *satisfy the subterm condition. Then* $\mathrm{in}_\delta \Phi$ *satisfies the subterm condition.*

THEOREM 1.15. *Let* L *be a strong language and let* Θ *be any finite set of nonvariable* L*-terms in which* v_0 *is the only variable to occur. There is an infinite set* Δ *of* L*-terms such that* Δ *satisfies the subterm condition and* $\Delta \perp \Theta$.

PROOF. For convenience we assume L has a binary operation symbol B. Let n be a natural number greater than the number of occurrences of symbols in any term in Θ. Let

$$\begin{array}{ll} \phi_0 = Bv_0B^n v_0^{n+1}, & \psi_0 = Bv_0Bv_0B^n v_0^{n+1}, \\ \phi_1 = Bv_0B^{n+1} v_0^{n+2}, & \psi_1 = Bv_0Bv_0B^{n+1} v_0^{n+2}, \\ \vdots & \vdots \\ \phi_n = Bv_0B^{2n} v_0^{2n+1}, & \psi_n = Bv_0Bv_0B^{2n} v_0^{2n+1}. \end{array}$$

Let $\pi = \phi_0[\phi_1[\phi_2 \cdots [\phi_n] \cdots]]$ and $\sigma = \psi_0[\psi_1 \cdots [\psi_n] \cdots]$. Based on a straightforward inspection we see that $\{\pi, \sigma\}$ satisfies the subterm condition. Suppose $\theta \in \Theta$ and ρ and η L-terms with $\theta[\eta] = \pi[\rho]$. From the structure of π it follows that there is a j with $0 < j \leq n$ such that $\phi_j[\cdots[\phi_n[\rho]]\cdots]$ is a subterm of η. Consequently η is longer than θ and so η can occur no more than n times as a subterm of $\theta[\eta]$. On the other hand $\phi_j[\cdots[\phi_n[\rho]]\cdots]$ occurs many more than n times in $\pi[\rho]$. This is a contradiction, so $\pi \perp \Theta$. The same can be said for σ. Now let $\Phi = \{f^2 g^{k+1} f g v_0 : k \in \omega\}$. Φ satisfies the subterm condition. Let $\delta(f) = \pi$ and $\delta(g) = \sigma$ and set $\Delta = \mathrm{in}_\delta \Phi$. By Theorem 1.14, Δ satisfies the subterm condition and moreover every member of Δ is a substitution instance of π. Hence $\Delta \perp \Theta$ and the proof is complete.

This theorem does not hold for languages which are not strong. Consider the language L_0 and let $\Theta = \{fv_0, gv_0, hv_0, kv_0\}$. There is no set Δ of L_0 with $\Delta \perp \Theta$. In fact we could take Θ to be any set consisting of all L_0-terms in v_0 whose length is less than a given $n > 1$ and the same would be true.

THEOREM 1.16 (THEOREM 2.9 IN [13]).

(*i*) *If* L *is a nontrivial language, then there is a denumerably infinite set* Δ *of* L*-terms which satisfies the subterm condition such that* v_0 *occurs in each of the terms in* Δ.

(*ii*) *If* L *is a strong language, then there is a set* Δ *of* L*-terms which satisfies the subterm condition such that for each* $n > 0$ *the variables* $v_0, \cdots, v_{n-1}$ *occur simultaneously in infinitely many terms in* Δ.

§**2. Undecidable properties.** In this section we will establish several of the undecidability results mentioned at the conclusion of the introduction. The proofs not included differ in detail and, to some extent, in conception from those presented here; but they do not differ in spirit and it is hoped that the interested reader will be able to devise for himself the proofs not included. It is not surprising that arguments establishing the undecidability of various properties of finite sets of equations in strong languages are simpler than those for the wider class of nontrivial languages. We begin this section with three theorems of a rather general nature that have between them most of the results mentioned in the introduction as special cases provided the language is strong. The rest of this section is devoted to obtaining theorems about nontrivial languages. Here I do not know any comprehensive general theorems and my approach to these results is more ad hoc. We begin with the single place in this paper where existential quantifiers are important.

DEFINITION 2.0. A first order sentence ϕ is a *Murskiĭ sentence* provided that ϕ is not universally valid and ϕ is a disjunction of existential prenex sentences, the quantifier-free part of each being an equation in which only one variable appears on each side.

THEOREM 2.1 (MURSKIĬ [14]). *Let L be a strong language. Let G be a collection of finite sets of L-equations with the following properties:*

(1) *if $\Gamma \in G$ and $\Gamma \equiv \Sigma$, then $\Sigma \in G$,*

(2) $\{v_0 \approx v_1\} \in G$, *and*

(3) *there is a Murskiĭ sentence ϕ such that $\Gamma \vdash \phi$ for all $\Gamma \in G$.*

Then G is not recursive.

PROOF. Let

$$\phi = \bigvee_{k<n} \exists v_0 v_1(\psi_k[v_0] \approx \theta_k[v_1]) \vee \bigvee_{j<m} \exists v_0(\sigma_j[v_0] \approx \tau_j[v_0])$$

be the Murskiĭ sentence specified in the theorem. Let Θ be the set of all nonvariable subterms of $\psi_k, \theta_k, \sigma_j$, and τ_j where $k < n$ and $j < m$. We can assume no variable different from v_0 occurs in any term in Θ. Since ϕ is not universally valid we observe that neither ψ_k nor θ_k may be v_0 and further that $\psi_k \neq \theta_k$ and $\sigma_j \neq \tau_j$ for all $k < n$ and $j < m$. According to Theorem 1.15 there must be four terms π_0, π_1, π_2, and π_3 such that $\{\pi_0, \pi_1, \pi_2, \pi_3\} \perp \Theta$ and $\{\pi_o, \pi_1, \pi_2, \pi_3\}$ satisfies the subterm condition. Let $\delta(f) = \pi_0$, $\delta(g) = \pi_1$, $\delta(h) = \pi_2$ and $\delta(k) = \pi_3$. Consider $B(M, \mu, \delta, \{v_0 \approx v_1\})$ for any equation μ in f, g, and v_0. According to Theorem 1.7

(i) If $M \vdash \mu$, then $B(M, \mu, \delta, \{v_0 \approx v_1\}) \equiv \{v_0 \approx v_1\}$ and therefore $B(M, \mu, \delta, \{v_0 \approx v_1\}) \in G$.

(ii) If $M \nvdash \mu$, then $B(M, \mu, \delta, \{v_0 \approx v_1\})$ has a denumerable model.

More can be said of (ii). If $M \nvdash \mu$, then M has a denumerable model in which μ fails. With the help of Theorem 1.3 $B(M, \mu, \delta, \{v_0 \approx v_1\})$ has a denumerable model $\mathfrak{A}$ such that $\psi_k^{\mathfrak{A}}$ and $\theta_k^{\mathfrak{A}}$ have disjoint ranges for all $k < n$ and moreover $\sigma_j^{\mathfrak{A}}$ and $\tau_j^{\mathfrak{A}}$ have disjoint ranges unless one is v_0 and in that case the other will have no fixed points, for $j < m$. Hence

(iii) if $M \nvdash \mu$, then $B(M, \mu, \delta, \{v_0 \approx v_1\})$ has a model in which ϕ fails and so $B(M, \mu, \delta, \{v_0 \approx v_1\}) \notin G$.

The theorem follows from (i) and (iii) by Theorem 1.8.

In [14] Murskiĭ formulates Theorem 2.1 for nontrivial languages. The proof he sketches is correct for strong languages and it is somewhat different from the proof just presented. The theorem is false for nontrivial languages as the following example reveals.

EXAMPLE 2.2. Let $\phi = \exists v_0 v_1[fv_0 = gv_1]$ and let $G = \{\Gamma : \Gamma \vdash \phi$ and Γ is a set of equations in f and $g\}$. Then $\Gamma \in G$ iff there is $\gamma \in \Gamma$ where f is the leftmost symbol of one side of γ and not the leftmost symbol of the other side. (If f is the leftmost symbol of both sides of every equation in Γ, then the two element model of Γ assigning f and g different constant functions will fail to satisfy ϕ.) This amounts to a decision procedure for G.

DEFINITION 2.3. Let L and L' be languages and let Δ be a set of L-equations while Θ is a set of L'-equations. We say that Δ is a *definitional reduct* of Θ iff there is an interpretation in_δ of L into L' such that for every infinite model $\mathfrak{A}$ of Δ there is a model $\mathfrak{B}$ of Θ with the same universe as $\mathfrak{A}$ and such that whenever Q is an operation symbol (of rank n) of L, we have $\delta(Q)^{\mathfrak{B}} = (Qv_0 \cdots v_{n-1})^{\mathfrak{A}}$.

We note that the notion of definitional reduct is closely connected to the notion of definitional (alias rational) equivalence of varieties. See Tarski [24] and especially Mal'cev [6].

THEOREM 2.4. *Let K be any collection of finite sets of equations in a strong language such that*

(i) $\{v_0 \approx v_1\} \in K$,

(ii) *if* $\Delta \in K$ *and* $\Gamma \equiv \Delta$, *then* $\Gamma \in K$, *and*

(iii) *there is a consistent set* Σ *of equations such that if* Σ *is a definitional reduct of* Δ, *then* $\Delta \notin K$.

Under these conditions K is not recursive.

PROOF. Let L be a strong language and let Σ be a set of equations fulfilling (iii). It does no harm to suppose that f, g, h, and k do not occur in Σ and that no operation symbol of rank 0 occurs in Σ. Let L' be the language whose operation symbols are those which occur in Σ together with f, g, h, and k. So L' has finitely many operation symbols. By Theorem 1.16, there will be an interpretation in_δ for L' into L with the range of δ satisfying the subterm condition. Let $\Delta(\mu) = B(M, \mu, \delta, \{v_0 \approx v_1\}) \cup \mathrm{in}_\delta\Sigma$ for each equation μ in f, g, and v_0.

(i) If $M \vdash \mu$, then $B(M, \mu, \delta, \{v_0 \approx v_1\}) \equiv \{v_0 \approx v_1\}$ and so $\Delta(\mu) \in K$.

(ii) If $M \nvdash \mu$ and $\mathfrak{A}$ is an infinite model of Σ, then M has a model $\mathfrak{B}$ with the same universe as $\mathfrak{A}$ such that μ fails in $\mathfrak{B}$. According to Theorem 1.7 (with an implicit use of Theorem 1.3) $\Delta(\mu)$ has a model $\mathfrak{C}$ with the same universe as $\mathfrak{A}$ establishing that Σ is a definitional reduct of $\Delta(\mu)$. Hence $\Delta(\mu) \notin K$.

Invoking Theorem 1.8 finishes the proof.

We note that the set Σ specified in Theorem 2.4 (iii) need not be in the same language as the sets of equations in K. But if Σ is in a nontrivial language then the language of K can be nontrivial, too.

THEOREM 2.5. *Let H be a collection of finite sets of equations in a strong language such that*

(i) *H is not empty,*

(ii) *if* $\Delta \in H$ *and* $\Delta \equiv \Gamma$, *then* $\Gamma \in H$, *and*

(iii) *for each* $\Gamma \in H$ *there is a term* τ *in which both* v_0 *and* v_1 *occur such that* $\Gamma \vdash \tau \approx v_0$.

Then H is not recursive.

PROOF. Let $\Gamma \in H$ and let τ be a nontrivial term such that $\Gamma \vdash \tau \approx v_0$. By Theorem 1.13, there are four terms ϕ_0, ϕ_1, ϕ_2, and ϕ_3 such that $\{\phi_0^+, \phi_1^+, \phi_2^+, \phi_3^+\}$ satisfies the subterm condition and $\Gamma \vdash \{\phi_0 \approx v_0, \phi_1 \approx v_0, \phi_2 \approx v_0, \phi_3 \approx v_0\}$. Let $\delta(f) = \phi_0$, $\delta(g) = \phi_1$, $\delta(h) = \phi_2$, and $\delta(k) = \phi_3$; $\delta^+(f) = \phi_0^+$, $\delta^+(g) = \phi_1^+$, $\delta^+(h) = \phi_2^+$, and $\delta^+(k) = \phi_3^+$. It is simple to verify that

(1) if $M \vdash \mu$, then $B(M, \mu, \delta, \Gamma) \equiv \Gamma$, so $B(M, \mu, \delta, \Gamma) \in H$. Also, by Theorem 1.7 (by an implicit use of Theorem 1.3) we obtain

(2) if $M \nvdash \mu$, then $B(M, \mu, \delta^+, \Gamma)$ has a model $\mathfrak{A}$ depending only on the operation symbols occurring in ϕ_0^+, ϕ_1^+, ϕ_2^+, and ϕ_3^+ such that $\mathfrak{A} \vdash \pi \approx v_0$ only if v_0 is the sole variable to occur in π.

By replacing all the unary operations of $\mathfrak{A}$ by the identity function we obtain a model $\mathfrak{B}$ of $B(M, \mu, \delta, \Gamma)$ with the same property. Hence $B(M, \mu, \delta, \Gamma) \notin H$. So the theorem is established by Theorem 1.8. (The reader should notice that any model of M can be extended to another model of M in which the operations have a common fixed point. This allows the use of Theorem 1.3(iii) in constructing the model $\mathfrak{A}$ above.)

Theorems 2.1 and 2.4 admit refinements. In fact it is possible to prove in these cases that the collection of singleton sets in G and the collection of those in K are not recursive. The key to this refinement is a theorem due to McKenzie and Tarski independently (see Tarski [24]). Their theorem asserts that for a certain finite set Γ of ring equations there is a recursive map F from finite sets of equations (regardless of language) into the set of all equations such that $F(\Gamma \cup \Sigma) \equiv \Gamma \cup \Sigma$.

The remaining three theorems of this paper are meant to illustrate how sharp results can be obtained for nontrivial languages. Perkins in [16] proved that in a language provided with two binary operations and two constants the property of being the base of a decidable theory is undecidable. We improve this theorem as follows.

THEOREM 2.6. *Let L be a recursive nontrivial language. Each of the following sets is not recursive*:

(i) $\{\Sigma : \Sigma$ *is a decidable set of L-equations*$\}$,

(ii) $\{\Sigma : \Sigma$ *is a decidable consistent set of L-equations*$\}$.

PROOF. Let L' be a nontrivial finite sublanguage of L. Let Γ be the set of equations asserting that all the operations have the same constant value. Then Γ is both consistent and decidable. Let θ any nontrivial term such that either no operation symbol of rank more than one occurs in θ or else the leftmost symbol in θ is an operation symbol of rank at least two. According to either Theorem 1.10 or Theorem 1.11 we can obtain a function δ from the operation symbols of L_0 into the terms of L' satisfying all the hypotheses of Theorem 1.7. So if μ and η are any equations in f, g, and v_0, then if $M \vdash \mu$, we conclude that $B(M, \mu, \delta, \Gamma) \equiv \Gamma$ and hence is both consistent and decidable and if $M \nvdash \mu$, then $M \vdash \eta$ iff $B(M, \mu, \delta, \Gamma) \vdash \mathrm{in}_\delta \eta$ and so $B(M, \mu, \delta, \Gamma)$ is consistent and undecidable. The proof is completed by invoking Theorem 1.8.

THEOREM 2.7. *Let L be any nontrivial language and let n be any positive integer. The following sets are not recursive.*

(i) $\{\Sigma: \Sigma$ *is an irredundant set of L-equations*$\}$,

(ii) $\{\Sigma: \Sigma$ *is a consistent irredundant set of L-equations*$\}$,

(iii) $\{\Sigma: \Sigma$ *is a set of L-equations with* $n \in \nabla\Sigma\}$,

(iv) $\{\Sigma: \Sigma$ *is a consistent set of L-equations with* $n \in \nabla\Sigma\}$.

PROOF. Consider first the language L' with just two operation symbols s and t, both unary. Let δ be the function defined by

$$\delta(f) = s^2tstv_0, \qquad \delta(g) = s^2t^2stv_0, \qquad \delta(h) = s^2t^3stv_0, \qquad \delta(k) = s^2t^4stv_0.$$

Let $\Delta = \{\delta(f), \delta(g), \delta(h), \delta(k), s^2t^5stv_0, s^2t^6stv_0\}$. Then Δ is a set of L'-terms satisfying the subterm condition. Let $\Gamma = \{sv_0 \approx tv_1\}$. Let μ be any equation in f, g, and v_0, and set $D(\mu) = B(M, \mu, \delta, \Gamma) \cup \{s^2t^5stv_0 \approx s^2t^6stv_0\}$.

Claim 1. If $M \vdash \mu$, then $D(\mu) \equiv \{sv_0 \approx tv_1\}$ and $D(\mu)$ is redundant.

Claim 2. If $M \nvdash \mu$, then $D(\mu)$ is irredundant and $1 \notin \nabla D(\mu)$.

PROOF. Recall that M is irredundant. Let f' and g' be new unary operation symbols and $N = M \cup \{f'v_0 \approx g'v_0\}$ and $\delta(f') = s^2t^5stv_0$ and $\delta(g') = s^2t^6stv_0$. Now N is irredundant and $D(\mu) = B(N, \mu, \delta, \Gamma)$. By Theorem 1.7, $D(\mu) \sim \{\mathrm{in}_\delta\eta\} \nvdash \mathrm{in}_\delta\eta$ whenever $\eta \in N$. Three equations remain to be checked in order to establish that $D(\mu)$ is irredundant. They have the forms $sv_0 \approx \alpha$, $tv_0 \approx \beta$, and $\gamma \approx \gamma'$ where v_0 and both s and t occur in α, β, and γ, and v_1 and both s and t occur in γ'. Since v_0 is the only variable occurring in $D(\mu) \sim \{\gamma \approx \gamma'\}$ we conclude that $D(\mu) \sim \{\gamma \approx \gamma'\} \nvdash \gamma \approx \gamma'$. Since no equation in $D(\mu) \sim \{sv_0 \approx \alpha\}$ has sv_0 as one of its sides we conclude $D(\mu) \sim \{sv_0 \approx \alpha\} \nvdash sv_0 \approx \alpha$. Similarly $D(\mu) \sim \{tv_0 \approx \beta\} \nvdash tv_0 \approx \beta$. So $D(\mu)$ is irredundant. Furthermore any base for $D(\mu)$ must have an equation with one side sy for some variable y, similarly one side must be ty and finally the base must include an equation with different variables on each side. So if $1 \in \nabla D(\mu)$ we would have $\{sv_0 \approx tv_1\} \equiv D(\mu)$. But by Theorem 1.7, $D(\mu) \nvdash sv_0 \approx tv_1$. This completes the proof of Claim 2.

Now let $\{\theta_0, \theta_1\} \cup \{\phi_i : 1 \leq i < n\} \cup \{\psi_i : 1 \leq i < n\}$ be any set of L-terms in which v_0 is the only variable to appear (and it occurs in every term) and such that the whole set satisfies the subterm condition. Let $\rho(s) = \theta_0$ and $\rho(t) = \theta_1$ and $E(\mu) = \mathrm{in}_\rho D(\mu) \cup \{\phi_i \approx \psi_i : 1 \leq i < n\}$.

Claim 3. If $M \vdash \mu$, then $E(\mu) \equiv \{\theta_0 \approx \theta_1[v_1]\} \cup \{\phi_i \approx \psi_i : 1 \leq i < n\}$ and $n \in \nabla E(\mu)$.

Claim 4. If $M \nvdash \mu$, then $E(\mu)$ is irredundant and $n \notin \nabla E(\mu)$.

PROOF. First observe that any base of $E(\mu)$ must include, up to renaming variables, $\{\phi_i \approx \psi_i : 1 \leq i < n\}$. (This is most easily seen by examining the possible proof of this set of equations.The subterm condition is important to this examination.) It follows from Claim 2 and Theorem 1.3 that $E(\mu)$ is irredundant. In addition to $\{\phi_i \approx \psi_i : 1 \leq i < n\}$ any base of $E(\mu)$ must include an equation with one side θ_0 (up to renaming variables), an equation with one side θ_1 (up to renaming again), and an equation in which two variables occur. Now by Theorems 1.7 and 1.3, $E(\mu) \nvdash \theta_0 \approx \theta_1[v_1]$. So $E(\mu)$ cannot have a base with n elements.

By Theorem 1.8, Claims 3 and 4 suffice to prove all parts of the theorem.

Theorem 2.7(iii) in the case when $n = 1$ and L has two binary operation symbols and two constants was found independently by D.Smith [22]. In the case $n = 1$ and L nontrivial, Theorem 2.7(iii) was announced in [11]. Don Pigozzi first proved Theorem 2.7(iii) in its full generality. The present proof differs from Pigozzi's proof which is unpublished. Theorem 2.7 answers some questions raised by Tarski in [24].

THEOREM 2.8. *Let L be a language with at least three unary operation symbols or some operation symbol of rank at least two. Let Δ be any set of L-equations. $\{\Sigma: \Sigma \vdash \Delta$ and Σ is a consistent set of L-equations$\}$ is recursive iff Δ has no consistent finitely based extensions.*

PROOF. If Δ has no consistent finitely based extensions, then the set in question is empty and hence recursive. We consider four disjoint cases to establish the converse.

Case I. Δ is true in every L-algebra. Then $\{\Sigma: \Sigma$ is a consistent set of L-equations$\} = \{\Sigma: \Sigma$ is a consistent set of L-equations and $\Sigma \vdash \Delta\}$. The conclusion follows by Theorem 3.15 of [13].

Case II. $\phi \approx \psi \in \Delta$ with $\phi \neq \psi$ and there is a consistent set Γ of L-equations and a nontrivial term θ in which all operation symbols are unary such that $\Gamma \vdash \Delta$ and $m(\theta) \cup \{\gamma: \gamma \approx \rho \in \Gamma$ or $\rho \approx \gamma \in \Gamma$ for some $\rho\}$ absorbs θ for Σ. Using Theorem 1.10 we obtain a map δ from the operation symbols of L_0 to terms in L which fulfills all the hypotheses of Theorem 1.7. So by Theorems 1.7 and 1.3 we obtain $M \vdash \mu$ iff $B(M, \mu, \delta, \Gamma) \vdash \Gamma$ iff $B(M, \mu, \delta, \Gamma) \vdash \phi \approx \psi$ for all equations μ in f, g, and v_0. The case is finished by appeal to Theorem 1.8.

Case III. $\phi \approx \psi \in \Delta$ with $\phi \neq \psi$ and there is a consistent set Γ of L-equations and a nontrivial term θ in which an operation symbol of rank at least two occurs such that $\Gamma \vdash \Delta$ and $m(\theta) \cup \{\gamma: \gamma \approx \rho \in \Gamma$ or $\rho \approx \gamma \in \Gamma$ for some $\rho\}$ absorbs θ for Γ. In the event that Δ does not fall into Case I or Case II and yet Δ^+ is true in every L-algebra, the proof is simple and left to the reader. So we assume $\phi^+ \neq \psi^+$. We use Theorem 1.13 in the same way we used Theorem 1.10 in the previous case to obtain $M \vdash \mu$ iff $B(M, \mu, \delta^+, \Gamma^+) \vdash \Gamma^+$ iff $B(M, \mu, \delta^+, \Gamma^+) \vdash \Delta^+$ iff $B(M, \mu, \delta^+, \Gamma^+) \vdash \phi^+ \approx \psi^+$ for all equations μ in f, g, and v_0. Let $N = \{sv_0 \approx v_0: s$ is a unary operation symbol of $L\}$. Clearly if $M \vdash \mu$, then $B(M, \mu, \delta^+, \Gamma^+) \cup N \vdash \Delta^+$ but $B(M, \mu, \delta^+, \Gamma^+) \cup N \vdash \{\rho \approx \rho^+: \rho$ is an L-term$\}$. Consequently $M \vdash \mu$ iff $B(M, \mu, \delta, \Gamma) \vdash \Delta$ and this case is done.

Case IV. None of the other cases hold. Let Γ be any finite consistent set of equations such that $\Gamma \vdash \Delta$ and let L' be the language specified by all the operation symbols occurring in Γ. The constant theory of L' (the theory asserting that all operations have the same constant value) cannot extend Δ unless one of the previous cases holds (see the proof of Theorem 2.6 for a typical use of the constant theory). Consequently, there must be an L'-term θ which is of the form $s^n v_0$ for some unary operation symbol s with $n > 0$ and $\theta \approx v_0 \in \Delta$. Let L'' be the language with all operation symbols of L' excepts s. Notice that L'' is nontrivial and let C be a base for the constant theory of L''. Evidently $C \cup \{sv_0 \approx v_0\} \vdash \Delta$. For any L-term θ let $\bar{\theta}$ be the term obtained by deleting all occurrences of s. Again the situation is simple if $\bar{\phi} = \bar{\psi}$ for all $\phi \approx \psi \in \Delta$. So we assume $\phi \approx \psi \in \Delta$ and $\bar{\phi} \neq \bar{\psi}$. By either Theorem 1.10 or

Theorem 1.11 we can find a function δ such that for every equation μ in f, g, and v_0

(1) if $M \vdash \mu$, then $B(M, \mu, \delta, \Gamma) \cup \{sv_0 \approx v_0\} \vdash \Delta$, and

(2) if $M \nvdash \mu$, then $B(M, \mu, \delta, \Gamma) \nvdash \bar{\phi} \approx \bar{\psi}$.

From (2) we easily obtain

(3) if $M \nvdash \mu$, then $B(M, \mu, \delta, \Gamma) \cup \{sv_0 \approx v_0\} \nvdash \phi \approx \psi$.

Hence $M \vdash \mu$ iff $B(M, \mu, \delta, \Gamma) \vdash \Delta$ and the proof of the claim finishes the theorem.

It should be remarked that this theorem fails to be true if L is permitted to have as operation symbols only two unary operation symbols. In particular $\{\Sigma : \Sigma \vdash \{fv_0 \approx v_0, gv_0 \approx gv_1\}$ and Σ is a set of equations in f and $g\}$ is recursive, cf. Example 4.5 in [13]. Independently, Murskiĭ announced in [14] a related result mentioned in the introduction.

§3. **Open problems.** At present there seems to be no satisfactory general theorem concerning undecidable properties of finite sets of equations in nontrivial languages. Also there is no general theorem known to me concerning properties not preserved under equivalence. I would be interested in work in both of these directions.

Here are problems of a more specific nature.

1. Let $m \geq n > 0$ and let $[n, m] = \{j : m \geq j \geq n$ and $j \in \omega\}$. Is $\{\Gamma : \nabla\Gamma = [n, m]$ and Γ is a set of L-equations$\}$ recursive for any nontrivial language and any integers $m \geq n > 0$?

2. Call a set Σ of L-equations *base-decidable* provided $\{\Gamma : \Gamma \equiv \Sigma\}$ is recursive. Is there any nontrivial language L such that $\{\Gamma : \Gamma$ is base-decidable and Γ is a set of L-equations$\}$ is recursive?

3. Discover some common algebraic or logical properties of finite sets of equations which turn out to be decidable.

4. Develop the theory of decidable properties of finite algebras. Many properties here appear to be decidable and analysis according to computational complexity would be of interest. It is not known whether the set of finitely based finite groupoids is recursive, cf. Perkins [15].

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