On the functional completeness of simple tournaments

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Abstract. The theory of multitraces provides a new proof that any simple tournament with more than two elements is functionally complete.

A *tournament* is a finite, directed, complete graph $\langle V; E\rangle$ without multiple edges. Write $x \to y$ to indicate that $x, y \in V$ and $(x, y) \in E$. In this paper tournaments have loops on all vertices, so $x \to x$ for all $x \in V$. Associate to a tournament $\langle V; E\rangle$ an algebra $\langle V; \cdot\rangle$ with the same universe and a binary product defined by $xy = x$ iff $x \to y$. Such an algebra is also called a tournament.

In [5], P. P. Pálfy applied Rosenberg's Completeness Theorem to prove that every simple tournament is functionally complete. Here we derive the same theorem from the theory of multitraces, [3], which is a part of tame congruence theory, [1].

A finite algebra $\mathbf{A}$ is *functionally complete* if every finitary operation on its universe is a polynomial of the algebra. A *trace* of a finite simple algebra $\mathbf{A}$ is a subset of A that is minimal among subsets $T \subseteq A$ satisfying $|T| > 1$ and $T = e(A)$ for some unary polynomial e satisfying $e(e(x)) = e(x)$. A *multitrace* of a finite simple algebra $\mathbf{A}$ is a subset $M \subseteq A$ such that $M = p(T, T, \ldots, T) = p(T^n)$ for some trace T and some n-ary polynomial p. It is known that if $\mathbf{A}$ is a finite simple algebra and T and T' are traces, then there are unary polynomials f and g such that $f(T) = T'$ and $g(T') = T$, so any trace can be used in the definition of "multitrace". It is also known that if T is a trace and f is a unary polynomial whose restriction to T is nonconstant, then $f(T)$ is another trace.

It is possible to construct an algebra on a trace $T = e(A)$ by equipping T with (the restrictions to T of) all operations of the form $e(p(\mathbf{x}))$, p a polynomial operation of $\mathbf{A}$. The result is called *the algebra* $\mathbf{A}$ *induces on* T, and is denoted $\mathbf{A}|_T$. It is shown in [1] that the algebras $\mathbf{A}|_T$ arising from different traces of $\mathbf{A}$ are polynomially equivalent algebras, and that they come in only five types, which are numbered $\mathbf{1} - \mathbf{5}$. Their polynomial equivalence types are: $\mathbf{1}$ = simple G-sets, $\mathbf{2}$ = 1-dimensional vector spaces, $\mathbf{3}$ = 2-element Boolean algebras, $\mathbf{4}$ = 2-element lattices, and $\mathbf{5}$ = 2-element semilattices.

The following specialization of Theorem 3.12 of [3] provides criteria for establishing functional completeness.

Theorem 1. *A finite algebra* $\mathbf{S}$ *is functionally complete if and only if*

(1) $\mathbf{S}$ *is simple of type* $\mathbf{3}$*, and*

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(2) *S is a multitrace.*

Lemma 2. *A simple tournament with more than two elements has type* **3**.

Proof. Theorem 33 of [2] states that any finite simple algebra in the variety generated by tournaments is itself a tournament. The proof starts with a short argument that all finite simple algebras in this variety have type **3**, **4**, or **5**. Most of the rest of the proof is an argument that this variety contains no simple algebra of type **5** that has more than two elements. This argument works for finite simple algebras of type **4** as well, and proves that there are no simple tournaments of type **4**. Hence any simple tournament with more than two elements has type **3**. □

Lemma 3. *Let* $\mathbf{S}$ *be a simple tournament with more than two elements.*

(1) $\mathbf{S}$ *contains a multitrace M and an element z such that $M \cup \{z\}$ is strongly connected and $|M \cup \{z\}| > 1$.*
(2) *If M is any multitrace of* $\mathbf{S}$ *and $M \cup \{z\}$ is strongly connected, then $M \cup \{z\}$ is also a multitrace.*
(3) *If M is a strongly connected multitrace and $1 < |M| < |S|$, then there is an element $z \in S - M$ such that $M \cup \{z\}$ is strongly connected.*

Proof. For (1), choose any trace $T = \{0, 1\}$ with elements labeled so that $0 \to 1$. Since $\mathbf{S}|_T$ is a 2-element Boolean algebra, there is a unary polynomial $p(x)$ inducing Boolean complementation on T. This polynomial restricted to T does not respect $\to$ in the sense that $0 \to 1$ but $p(0) \not\to p(1)$. The constant polynomials and the identity polynomial do respect $\to$ in this sense, so there must exist a unary polynomial $f(x) = g(x) \cdot h(x)$ such that $g(0) \to g(1)$ and $h(0) \to h(1)$, but $f(0) \not\to f(1)$. Since $f(0) = g(0) \cdot h(0)$ we have $f(0) \in \{g(0), h(0)\}$, and similarly $f(1) \in \{g(1), h(1)\}$, but $(f(0), f(1)) \neq (g(0), g(1))$ or $(h(0), h(1))$ since f does not preserve $\to$ while both g and h do. Hence $(f(0), f(1)) = (g(0), h(1))$ or $(h(0), g(1))$. The cases are symmetric, so consider the case $(f(0), f(1)) = (g(0), h(1))$, which is depicted in Figure 1.

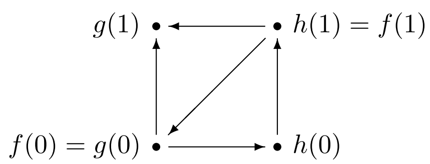

Figure 1

The directions on the nonhorizontal arrows follow from the assumptions that $0 \to 1$ and g and h respect $\to$ while f does not. The directions on the horizontal arrows follow from $g(0) = f(0) = g(0) \cdot h(0) \to h(0)$ and $h(1) = f(1) = g(1) \cdot h(1) \to g(1)$. Significantly, $\{g(0), h(0), h(1)\}$ is a directed $\to$-cycle, implying that these three elements are distinct. Since $M := \{h(0), h(1)\}$ is a 2-element image of a trace T under a polynomial, M is a (multi)trace. For $z = g(0)$ we get that our directed $\to$-cycle is $M \cup \{z\}$, which is strongly connected and contains more than one element.

For (2), note that if $A = p(T^m)$ and $B = q(T^n)$ are multitraces, then the complex product $AB = \{ab \mid a \in A, b \in B\}$ is also a multitrace, since $AB = r(T^{m+n})$ for $r(\mathbf{xy}) = p(\mathbf{x})\cdot q(\mathbf{y})$. Moreover, any singleton set is a multitrace, being the image of a constant unary polynomial. Thus, if M is a multitrace, so are the complex products $M\{z\}, M(M\{z\}), M(M(M\{z\}))$, etc. We argue that this is an increasing sequence of sets which terminates at $M \cup \{z\}$ whenever $M \cup \{z\}$ is strongly connected.

Since $M \cup \{z\}$ is strongly connected, there exists $m \in M - \{z\}$ such that $z \to m$, equivalently $z = mz$. Thus, $\{z\} \subseteq M\{z\}$. Multiplying both sides of this inclusion by M repeatedly yields $M\{z\} \subseteq M(M\{z\}) = M^2\{z\}$, then $M^2\{z\} \subseteq M^3\{z\}$, etc. Thus the multitraces $M^i\{z\}$ increase with i. They are contained in $M \cup \{z\}$ since this set is a subalgebra of $\mathbf{S}$. If $X := \bigcup_i M^i\{z\}$, then $X = M^j\{z\}$ for some large j, which makes X a multitrace. By construction we have $MX = X$, so there is no directed edge from $M - X$ into X. Since $z \in X$, there can be no directed edge from the smaller set $(M \cup \{z\}) - X$ into X either. But $M \cup \{z\}$ is strongly connected and X is a nonempty subset, so this forces $M \cup \{z\} = X =$ a multitrace.

For (3), we use the simplicity criterion for tournaments from [4] (Proposition 4): a tournament $\mathbf{S}$ is simple iff for every subset M satisfying $1 < |M| < |S|$ there is an element $z \in S - M$ and elements $a, b \in M$ such that $a \to z \to b$. This produces the element z we need in (3): M is strongly connected and z is connected to and from M through a and b, so $M \cup \{z\}$ is also strongly connected. □

Items (1) and (2) of this lemma produce a nontrivial strongly connected multitrace, while items (2) and (3) allow one to grow this multitrace without restriction until we reach S. Together with Theorem 1 and Lemma 2, we get the desired result.

Theorem 4. *A simple tournament with more than two elements is functionally complete.*

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